

# Quirky Quantum Updates

## The Anti-Textbook\*

Updates to Quirky Quantum Concepts:

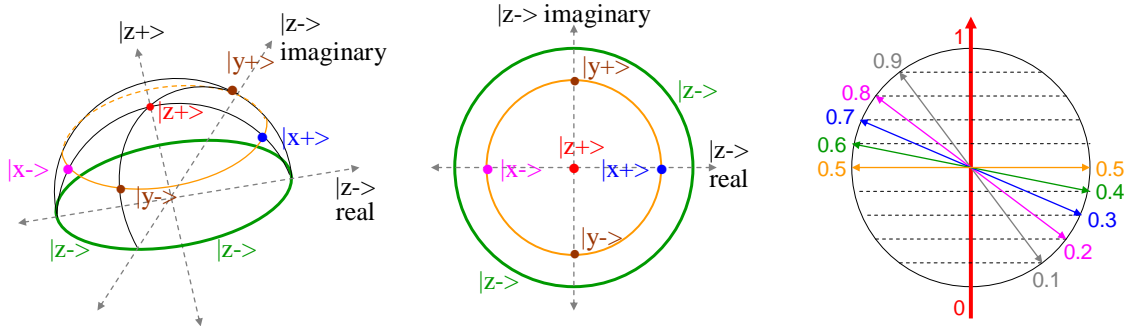
Buy the original locally at the UCSD bookstore: [Quirky Quantum Concepts](#)

From Springer: [Quirky Quantum Concepts](#)

From Amazon: [Quirky Quantum Concepts](#)

A Work In Progress. See [physics.ucsd.edu/~emichels](http://physics.ucsd.edu/~emichels) for the latest versions of the Funky Series.  
Please send me comments.

### Eric L. Michelsen



“Quantum Mechanics is a silly theory, perhaps the silliest theory to come out of the 20<sup>th</sup> century. The only reason it has any following at all is that it is completely supported by experiment.” – *Unknown physicist*

“We are all agreed that your theory is crazy. The question that divides us is whether it is crazy enough to have a chance of being correct.” – *Niels Bohr*

“Now in the further development of science, we want more than just a formula. First we have an observation, then we have numbers that we measure, then we have a law which summarizes all the numbers. But the real *glory* of science is that we can *find a way of thinking* such that the law is *evident*.” – *Richard Feynman*

\* Physical, conceptual, geometric, and pictorial physics that didn't fit in your textbook.

Instead of distributing this document, please link to [physics.ucsd.edu/~emichels/QuirkyQuantumUpdates.pdf](http://physics.ucsd.edu/~emichels/QuirkyQuantumUpdates.pdf).  
Please cite as: Michelsen, Eric L., *Quirky Quantum Updates*, [physics.ucsd.edu/~emichels](http://physics.ucsd.edu/~emichels), 2/24/2020.

**Physical constants:** 2006 values from NIST. For more, see <http://physics.nist.gov/cuu/Constants/>.

Speed of light in vacuum	$c = 299\,792\,458\text{ m s}^{-1}$ (exact)
Gravitational constant	$G = 6.674\,28(67) \times 10^{-11}\text{ m}^3\text{ kg}^{-1}\text{ s}^{-2}$
Relative standard uncertainty	$\pm 1.0 \times 10^{-4}$
Boltzmann constant	$k = 1.380\,6504(24) \times 10^{-23}\text{ J K}^{-1}$
Stefan-Boltzmann constant	$\sigma = 5.670\,400(40) \times 10^{-8}\text{ W m}^{-2}\text{ K}^{-4}$
Relative standard uncertainty	$\pm 7.0 \times 10^{-6}$
Avogadro constant	$N_A, L = 6.022\,141\,79(30) \times 10^{23}\text{ mol}^{-1}$
Relative standard uncertainty	$\pm 5.0 \times 10^{-8}$
Molar gas constant	$R = 8.314\,472(15)\text{ J mol}^{-1}\text{ K}^{-1}$
calorie	4.184 J (exact)
Electron mass	$m_e = 9.109\,382\,15(45) \times 10^{-31}\text{ kg}$
Proton mass	$m_p = 1.672\,621\,637(83) \times 10^{-27}\text{ kg}$
Proton/electron mass ratio	$m_p/m_e = 1836.152\,672\,47(80)$
Elementary charge	$e = 1.602\,176\,487(40) \times 10^{-19}\text{ C}$
Electron g-factor	$g_e = -2.002\,319\,304\,3622(15)$
Proton g-factor	$g_p = 5.585\,694\,713(46)$
Neutron g-factor	$g_N = -3.826\,085\,45(90)$
Muon mass	$m_\mu = 1.883\,531\,30(11) \times 10^{-28}\text{ kg}$
Inverse fine structure constant	$\alpha^{-1} = 137.035\,999\,679(94)$
Planck constant	$h = 6.626\,068\,96(33) \times 10^{-34}\text{ J s}$
Planck constant over $2\pi$	$\hbar = 1.054\,571\,628(53) \times 10^{-34}\text{ J s}$
Bohr radius	$a_0 = 0.529\,177\,208\,59(36) \times 10^{-10}\text{ m}$
Bohr magneton	$\mu_B = 927.400\,915(23) \times 10^{-26}\text{ J T}^{-1}$

## Contents

<b>1</b>	<b>Basic Wave Mechanics Concepts .....</b>	<b>4</b>
1.1.1	Another example of Quantum State Evolution Through Measurements .....	4
1.2	Thinking Quantumly .....	4
<b>2</b>	<b>Riding the Wave: More on Wave Mechanics .....</b>	<b>7</b>
2.1	Kinky Variations .....	7
2.1.1	Exponential Trial Wave-function .....	8
2.1.2	Another Kinky Variation: Rational Function.....	9
2.1.3	The Exact Solution.....	11
2.1.4	A Smooth Variation .....	12
2.1.5	A Less Steep Rational Function.....	13
2.1.6	Triangles on Trial .....	14
2.1.7	How Steep Is Steep? .....	15
2.2	The WKB Approximation.....	17
2.2.1	WKB For Tunneling .....	17
<b>3</b>	<b>Desultory Topics in Quantum Mechanics .....</b>	<b>19</b>
3.1	Unobserved Things Affect Observed Things .....	19
<b>4</b>	<b>Quantum Electromagnetic Radiation .....</b>	<b>20</b>
4.1	Quantum “Fluctuations” and Electromagnetic Zero Point Energy.....	20
4.1.1	Introduction.....	20
4.1.2	Definitions, and Statement of the Issue .....	20
4.1.3	Simple Proof That EM ZPE is Unobservable .....	22
4.1.4	Burden of Proof: Are Vacuum Fluctuations a Credible Claim in the First Place?.....	24
4.1.5	That’s Intense: Photon Absorption .....	24
4.1.6	Spontaneous Emission .....	25
4.1.7	The Lamb Shift.....	25
4.1.8	Casimir Force Is Not Due to Vacuum Energy .....	28
4.1.9	Conclusions on EM Vacuum Fields .....	31
4.1.10	References.....	32
<b>5</b>	<b>Appendices .....</b>	<b>34</b>
5.1	References.....	34
5.1.1	Books.....	34
5.1.2	Articles.....	34

# 1 Basic Wave Mechanics Concepts

## 1.1.1 Another example of Quantum State Evolution Through Measurements

Consider light passing through a sequence of 3 polarizers: vertical, 45-degree, and horizontal. The probabilities are only collapsed after the final observation (or lack) of a photon. We can start after the vertical polarizer, and consider a photon of definite vertical polarization. After passing the 45-deg polarizer, the photon is in a superposition of absorbed and 45-deg polarization:

$$|\psi\rangle = \frac{1}{\sqrt{2}}|abs_1\rangle + \frac{1}{\sqrt{2}}|45^\circ\rangle.$$

After the horizontal filter (but before observation), the photon is in the state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}|abs_1\rangle + \frac{1}{2}|abs_2\rangle + \frac{1}{2}|horizontal\rangle.$$

In this case, observing is a partial measurement, because lack of detecting a photon doesn't say which polarizer absorbed it. The probability of observing the photon is  $(1/2)^2 = 1/4$  (and if observed, it will be horizontal). If you don't observe it (and you have a perfect detector with 100% quantum efficiency), the state is a normalized version of the first two terms:

$$|\psi\rangle = \sqrt{\frac{2}{3}}|abs_1\rangle + \sqrt{\frac{1}{3}}|abs_2\rangle.$$

---

## 1.2 Thinking Quantumly

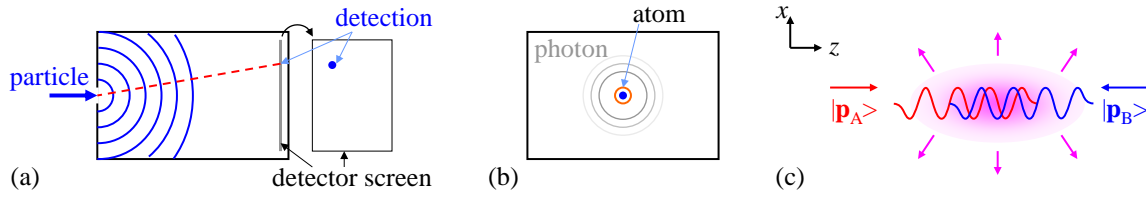
To be uncertain is to be uncomfortable, but to be certain is to be ridiculous. – Chinese Proverb, [proverbicals.com/certainty].

To quickly assess a quantum situation, and qualitatively predict its behavior, one must develop quantum insight, i.e., learn to “think quantumly.” We briefly introduce here some necessary fundamental concepts. We discuss measurement theory and other topics in more detail in later sections.

No event actually happens in QM, until you make an observation. After an observation, we tend to imagine that a single sequence of events led to the observation; this is not true. Time evolution creates quantum state components for *all* events leading to *all* possible outcomes. We know this because some outcomes interfere with others before the observation. The danger with thinking of a single sequence of events is that it suggests that all these outcomes are somehow fluctuating amongst themselves (they are not). As noted [elsewhere], quantum time evolution is smooth and continuous; there are no fluctuations, and no discontinuities. The fluctuation myth also leads to imagining violations of conservation of energy that are not actually there.

For example, consider a system in a stationary state; it is just sitting there, waiting to be measured. There are many possible outcomes of the measurement. *All* of those outcomes are *already* present in the current stationary quantum state, each with its own amplitude. Sometime in the past, the system time evolved into its current stationary state; that time evolution created all the state components for all possible outcomes. Let us consider some specific examples.

**Quantum Hindsight: Yesterday's News:** Consider a particle squeezing through a single, narrow slit, narrow enough that there is only a central diffraction spread, with no minima (Figure 1.1a). We feed in a particle of well-measured energy, so we also know the magnitude of momentum. The particle is soon detected on a screen. When we observe the detection, we know the particle's position arbitrarily well. After observation, we might imagine that we can reconstruct the “path” the particle took, and exactly its momentum vector just before impact on the detector. In hindsight, we might say that we *now* know exactly the particle's position and momentum at the (past) event of detection, with no uncertainty. But a better way to think about it is to say that a particle with that exact position and momentum was a *component* of the quantum state just before detection, and by chance, that component was selected by our observation. Note that before our observation, there were components of detection all over the screen.



**Figure 1.1** (a) A particle diffracts through a slit and is detected at a specific point. (b) Excited atom in a box. (c) Two particles scattering.

Of course, if the particle survived the detection (say an electron, rather than a photon), then its *current* position and momentum are still subject to uncertainty. The uncertainty principle is not about knowledge; it’s about reality itself (see *The Real Uncertainty Principle* [elsewhere]). A particle, in any given moment, cannot *have* a well-defined position and momentum. The Schrodinger equation forbids it. But after the fact, there is no limit to how well we can measure the position and momentum it *used to have*. [We’ve heard this example attributed to Feynman, but have no reference for it.]

**Decay:** Consider an excited atom in a box, Figure 1.1b. We wait a while, and then peek inside to see if the atom has decayed and radiated a photon. We might imagine that during our wait, the atom either decays or it doesn’t, and when we peek, we will see which. But that is *not* how quantum time evolution works. In fact, the quantum state is smoothly changing with time, and the atom never “decays” in the sense of a sudden event. During the wait, the quantum state is an evolving entangled state of the atom and EM field state, and varies over time like this:

$$|\psi(t)\rangle = a(t) \underbrace{|exc\rangle}_{\text{atom}} \underbrace{|N=0\rangle}_{\text{EM field}} + b(t) |gnd\rangle |N=1\rangle, \quad \text{where } a(0)=1, b(0)=0, a(t \rightarrow \infty)=0, b(t \rightarrow \infty)=1.$$

When we observe the quantum state, we see either the atom is still excited, or it is in its ground state with a photon in the box. For longer waits, the probability of seeing the atom excited goes to 0, and seeing it decayed goes to 1.

Note that to have such a superposition of photon states with different photon numbers, the EM field *must* be quantized. A stronger example is the Lamb shift, where the atom’s state includes both no-photon and photon components [Be&Sa 1957 p97b, there should be a better one??]. This shift can only be explained with a quantized EM field.

**Scattering:** Two particles with long wave-packets collide head-on along the z-axis, Figure 1.1c. They scatter, i.e., while their wave-packets overlap, they time-evolve into a nearly stationary state; the only time evolution left is that the outbound particles (really, quantum state components) travel outward from the collision region. After scattering, the quantum state is a big entangled set of components, one component for each possible outcome. The state might be:

$$|final\rangle = c_1 |p_{A1}\rangle |p_{B1}\rangle + c_2 |p_{A2}\rangle |p_{B2}\rangle + \dots + c_j |p_{Aj}\rangle |p_{Bj}\rangle + \dots + c_k |other\ stuff\rangle + \dots$$

$$\text{where } p_{Aj} + p_{Bj} = \mathbf{0}_v \quad \forall j.$$

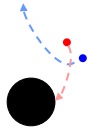
The final state is a superposition of all possible scattering directions. Different components of the final state may have different numbers of particles. Eventually, the particles are recorded by detectors (which decohere the state components, turning them into classical probabilities). In the decoherence model [elsewhere], a mini-collapse occurs when an observer reads the detectors, selecting one outcome. At this point, we’re tempted to say the particles scattered into this specific observed outcome. It is much better to recognize that they scattered into a superposition of all possible outcomes, and our observation selected one of those.

After a measurement, it is somewhat valid to reconstruct a sequence of events leading up to the measurement: the *component* selected by our measurement did, in fact, follow that sequence. Just before we observe, the quantum state *simultaneously* reflects all the chains of events leading to all possible outcomes. After measuring, it is valid to consider the one chain that the measurement selected.

**Black hole radiation:** The common myth about black holes radiating by absorbing a “negative-energy” particle is completely wrong. A full treatment is complicated, and computes the full density matrix of particles just outside the event horizon, and also at infinity [ref??]. We give here a simple, but plausible, model.

First, note that a black hole is a quasi-stationary state; it can't be stationary, because it's evaporating. Before a radiating particle is detected, the quantum state includes an outcome for that radiating particle. This outcome is a sequence of events starting with an earlier quantum state that included a component with a particle-antiparticle pair (Figure 1.2a). This pair is part of the original energy of the black hole; it is *not* created by popping into existence out of nothing. Instead, it is created by the time evolution of the hole itself. From there, the state component evolves: the particles scatter off each other, under the influence of the black hole's gravity. It is well-known, even in classical gravity scattering, that such a 3-body interaction can eject one particle from the gravitational well of the system, while conserving total energy. In other words, the blue particle can acquire escape speed from the interaction.

The kinetic energy needed to escape comes from the gravitational potential energy lost by the red particle falling in, which was transferred to the blue particle through the red/blue interaction. Energy is conserved all the time, with no time-uncertainty, and no quantum fluctuations. The energy carried away by the blue particle was part of the black hole system to start with. After the radiation, the black hole system has lost the energy carried away. There are an infinite number of particle-antiparticle components in the black hole state, new ones get created continuously, and some radiate to infinity, as described above. Through this process, the black hole quantum state evolves into a near certainty that all of the black hole has evaporated away.



(a) black hole

**Figure 1.2** (a) Particle-antiparticle pair outside a black hole. As in classical gravitational scattering, it is possible for one particle to be ejected from the system, and eventually detected.

---

**Is “quantumly” a word?** Symmetry is an important concept in physics. If it is true that any noun can be verbed, then by symmetry, any adjective can be adverbized. Similarly, if one can “think classically,” then symmetry demands that one can also “think quantumly.”

## 2 Riding the Wave: More on Wave Mechanics

### 2.1 Kinky Variations

The variational principle provides a fairly simple way to estimate the ground state energy of a system that we cannot solve exactly. We often hear that our choice of trial wave-function isn't very important; the variational principle works "pretty well" with most any trial wave-function [Gri ??, others]. However, we show here that making a judicious choice of trial wave-function can significantly improve the accuracy of your estimate. We compare here 6 trial wave-functions: the cosine, a symmetric exponential, two symmetric rational functions, a triangle, and a gaussian.

You may have seen a variational problem to estimate the ground state energy for a delta-function potential,

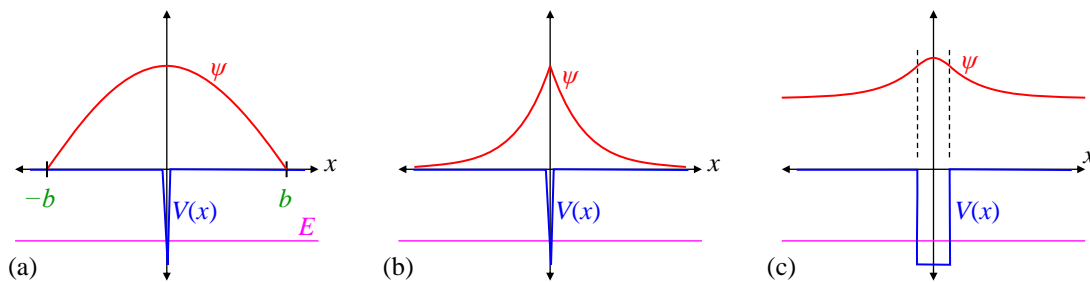
$$V(x) = -\alpha\delta(x), \quad \alpha > 0, [\alpha] = \text{J}\cdot\text{m},$$

using a cosine half-cycle as the trial wave-function (Figure 2.1a).

A delta-function can be a reasonable approximation to some tight potentials where the decay length of  $\psi$  is much longer than the width of the potential well. For example, in the early days of nuclear physics, experimentalists used a delta-function potential to approximate the binding energy of nuclei. This estimate was a crucial part of the design of the hardware used to measure the nuclear binding energy.

The cosine wave-function is not very good, being off by a factor of  $\sim 2.5$  in ground-state energy [Gri chap 7??]. Could we have reasonably done better?

You're darn tootin'.



**Figure 2.1** (a) A poor trial wave-function. (b) A better trial wave-function. (c) Close-up of infinitesimal behavior near the delta-function.

We compare here several trial wave-functions, with these results:

Trial Function	Error
exponential	0
$(bx + 1)^{-n}$ (two-parameter)	0
$(bx + 1)^{-2}$	6%
$(bx + 1)^{-1}$	25%
triangle	25%
gaussian	57%
cosine	150%

The exponential happens to be exact, and the best 1-parameter rational function shown above is within 6%. The smooth functions, cosine and gaussian, are off by factors of  $\sim 2.5$  and 1.5. Finally, we minimize a two-parameter trial rational function, and find it also yields the exact answer (probably by coincidence). We conclude:

The qualitative features of the trial wave-function are important to getting an accurate estimate of the ground state energy with the variational principle.

What qualitative features might we put into our trial wave-functions? First, we expect  $|\psi|^2$  to be concentrated near the attractive potential. While  $\cos(x)$  has this feature, it is broad and flat near the delta-function, as is the gaussian. A sharper concentration is likely better. Second, it is a general feature of the Schrodinger Equation (SE) that a delta function in the potential causes a discontinuity in  $\psi'$ , i.e. a “kink.” Therefore, a wave-function like Figure 2.1b is a better choice.

We note that the entire variational principle is usually justified on the completeness of the eigenfunctions: any arbitrary function can be written as a superposition of eigenfunctions. This insures the ground state energy is the minimum of all possible energies. This proof works only for complete eigenfunction sets. For finite potential wells, of which the delta-function potential is a limiting case, the bound-state eigenfunctions are *not* complete. The delta-function potential has only one bound state, and the finite well has only a finite number of bound states. It's impossible to construct an arbitrary function from only a finite number of basis functions. Therefore, we have no reason to expect that the variational principle works at all for such potentials. Nonetheless, it *does* seem to work. We proceed cautiously, as we may be on shaky ground.

### 2.1.1 Exponential Trial Wave-function

One candidate trial wave-function that looks like Figure 2.1b is:

$$\psi(x) = Ae^{-b|x|}, \quad b > 0.$$

(One can see almost immediately that this is the exact wave-function:  $\psi$  in a classically forbidden region of constant potential is always a decaying exponential. More below.) Our variational parameter is  $b$ . We now follow the usual steps for minimizing the variational energy [Gri ch. 7]:

- find the normalization factor  $A$  in terms of  $b$ ;
- compute  $\langle H \rangle = \langle T \rangle + \langle V \rangle$  as functions of  $b$ ;
- find the  $b$  that minimizes  $\langle H \rangle$ ;
- and finally, find the minimum  $\langle H \rangle$ .

**Normalize:**

$$\int_{-\infty}^{\infty} \psi^*(x)\psi(x) dx = 1 = A^2 \left[ \int_{-\infty}^0 e^{+2bx} dx + \int_0^{\infty} e^{-2bx} dx \right] = A^2 \left[ 2 \int_0^{\infty} e^{-2bx} dx \right] = -A^2 \frac{2}{2b} \left[ e^{-2bx} \right]_0^{\infty} = \frac{A^2}{b}$$

$$A^2 = b.$$

Note that  $b$  has units of  $m^{-1}$ , and  $[A] = [\psi] = m^{-1/2}$ , as necessary.

**Compute  $\langle H \rangle = \langle T \rangle + \langle V \rangle$ :** The average potential energy is straightforward from the definition of the delta-function, and its integral:

$$\langle V \rangle = \int_{-\infty}^{\infty} \psi^*(x)V(x)\psi(x) dx = -\alpha A^2 \int_{-\infty}^{\infty} e^{+bx} \delta(x) e^{-bx} dx = -b\alpha, \quad \text{using } A^2 = b.$$

The kinetic energy is a little bit subtle, because we must take the 2<sup>nd</sup> derivative of  $\psi$ , which is infinite at  $x = 0$  (the kink). Therefore, we split the integral into 3 parts:

$$\langle T \rangle = \int_{-\infty}^{\infty} \psi^*(x) \left( \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right) \psi(x) dx$$

$$= \frac{-\hbar^2}{2m} \left[ A^2 \int_{-\infty}^{0^-} e^{+bx} b^2 e^{+bx} dx + \int_{0^-}^{0^+} \psi^*(x) \frac{\partial^2}{\partial x^2} \psi(x) dx + A^2 \int_{0^+}^{\infty} e^{-bx} b^2 e^{-bx} dx \right].$$

The first and third integrals are simple, and equal:



$$I \equiv A^2 \int_{-\infty}^0 e^{+bx} b^2 e^{+bx} dx = b^3 \left[ \frac{1}{2b} e^{2bx} \right]_{-\infty}^0 = \frac{b^2}{2}.$$

The middle integral can be evaluated from the Fundamental Theorem of Calculus, to wit: the integral of a derivative equals the difference of the original function at the endpoints:

$$\int_a^b f'(x) dx = f(b) - f(a) \Rightarrow \int_a^b \psi'' dx = \psi'(b) - \psi'(a).$$

Then the middle integral becomes:

$$\psi'(x) = \pm A b e^{\pm bx} \Rightarrow \int_{0^-}^{0^+} \psi''(x) dx = \psi'(0^+) - \psi'(0^-) = A^2 (-be^0 - be^0) = -2b^2.$$

Physically, we can visualize the wave-function to be as in Figure 2.1c, in the limit as the near-delta-function goes to zero width. Note that in this transition region, the KE is positive ( $\psi$  is concave toward the axis).

Combining the 3 terms:

$$\langle T \rangle = \frac{-\hbar^2}{2m} (I - 2b^2 + I) = +\frac{\hbar^2 b^2}{2m}.$$

Note that  $\langle T \rangle$  is positive, as it must be on physical grounds. Mathematically, for square-integrable functions (which wave-functions must be),  $\hat{T}$  is a positive definite operator. This holds true even though the kinetic energy is negative for all of the wave-function, except at the discontinuity (where the local kinetic energy is infinite over an infinitesimal distance).

Finally:

$$\langle H \rangle = \frac{\hbar^2 b^2}{2m} - b\alpha.$$

**Find the  $b$  that minimizes  $\langle H \rangle$ :**

$$\frac{d\langle H \rangle}{db} = 0 = \frac{\hbar^2 b}{m} - \alpha, \quad b = \frac{m\alpha}{\hbar^2}. \quad (2.1)$$

**Find the minimum  $\langle H \rangle$ :**

$$\langle H \rangle_{\min} = \left[ \frac{\hbar^2 b^2}{2m} - b\alpha \right]_{b=m\alpha/\hbar^2} = \frac{\hbar^2}{2m} \left( \frac{m^2 \alpha^2}{\hbar^4} \right) - \frac{m\alpha^2}{\hbar^2} = \frac{m\alpha^2}{2\hbar^2} - \frac{m\alpha^2}{\hbar^2} = -\frac{m\alpha^2}{2\hbar^2}. \quad (2.2)$$

Since we happened to guess the exact wave-function, this is the exact ground-state energy.

### 2.1.2 Another Kinky Variation: Rational Function

Our first guess for the variational wave-function turned out to be the exact wave-function. But how well would a different trial wave-function do? Let us try another “good-looking” wave-function, whose graph is qualitatively the same as before (Figure 2.1b):

$$\psi(x) = A \frac{1}{(b|x|+1)^2}, \quad b > 0.$$

It satisfies the requirement for a kink at  $x = 0$ , and is normalizable. Note that ‘ $b$ ’ here is completely different from our previous ‘ $b$ ’, though it still has units of  $m^{-1}$ . We now follow the standard four steps for the variational estimate of the ground state energy. The algebra is a little more involved than before, but straightforward. Note that our result will include dimensionless constant factors; we must compute these rigorously, because the accuracy of our result depends on them. This includes scrupulous computing and use of the normalization factor  $A$ .

**Normalize:** We use the even-ness of  $\psi$ , so we double the right-half integral:

$$1 = 2A^2 \int_0^\infty (bx+1)^{-4} dx = -\frac{2A^2}{3b} \left[ (bx+1)^{-3} \right]_0^\infty = \frac{2A^2}{3b}, \quad A^2 = \frac{3b}{2}.$$

**Compute  $\langle H \rangle = \langle T \rangle + \langle V \rangle$ :** The average potential energy is straightforward from the definition of the delta-function, and its integral:

$$\langle V \rangle = \int_{-\infty}^\infty \psi^*(x)V(x)\psi(x) dx = -\alpha A^2 \int_{-\infty}^\infty (1)\delta(x)(1) dx = -\frac{3b}{2}\alpha, \quad \text{using } A^2 = \frac{3b}{2}.$$

The kinetic energy is a little bit subtle, because we must take the 2<sup>nd</sup> derivative of  $\psi$ , which is infinite at  $x = 0$  (the kink), so we split up the integral:

$$\langle T \rangle = \int_{-\infty}^\infty \psi^*(x) \left( \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right) \psi(x) dx = \frac{-\hbar^2}{2m} \left[ \int_{0^-}^{0^+} \psi^*(x)\psi''(x) dx + 2 \int_0^\infty \psi^*(x)\psi''(x) dx \right].$$

The derivatives of  $\psi$  are:

$$\begin{aligned} \psi'(x > 0) &= Ab(-2)(bx+1)^{-3}, & \psi'(x < 0) &= Ab(+2)(-bx+1)^{-3} \\ \psi''(x) &= Ab^2(+6)(b|x|+1)^{-4} \end{aligned}$$

As before, we use the Fundamental Theorem of Calculus for the first integral:

$$\begin{aligned} \langle T \rangle &= \frac{-\hbar^2}{2m} \left[ \int_{0^-}^{0^+} \psi^*(x) \frac{\partial^2}{\partial x^2} \psi(x) dx + 2A^2 b^2 \int_0^\infty (bx+1)^{-2} 6(bx+1)^{-4} dx \right] && \text{Use } 2A^2 = 3b \\ &= \frac{-\hbar^2}{2m} \left\{ \psi^*(0) (\psi'(0^+) - \psi'(0^-)) + \frac{18b^2}{(-5)} \left[ (bx+1)^{-5} \right]_0^\infty \right\} && (2.3) \\ &= \frac{-\hbar^2}{2m} \left\{ A^2 b (-2 - 2) - \frac{18}{5} b^2 (-1) \right\} \\ &= \frac{-\hbar^2}{2m} \left\{ -6b^2 + \frac{18}{5} b^2 \right\} = \frac{+\hbar^2}{m} \left( \frac{6}{5} \right) b^2. \end{aligned}$$

As before, we can visualize the wave-function to be as in Figure 2.1c, in the limit as the near-delta-function goes to zero width. Note that in this transition region, the KE is positive ( $\psi$  is concave toward the axis). Also,  $\langle T \rangle$  is positive, as it must be on physical grounds. Mathematically, for square-integrable functions,  $\hat{T}$  is a positive definite operator. This holds true even though the kinetic energy is negative for all of the wave-function, except at the discontinuity (where kinetic energy is infinite).

Finally:

$$\langle H \rangle = \frac{\hbar^2}{m} \left( \frac{6}{5} \right) b^2 - \frac{3b}{2} \alpha.$$

**Find the  $b$  that minimizes  $\langle H \rangle$ :**

$$0 = \frac{d\langle H \rangle}{db} = \frac{12\hbar^2}{5m} b - \frac{3\alpha}{2}, \quad b = \frac{5m\alpha}{8\hbar^2}.$$

Remember, this  $b$  has nothing to do with  $b$  from our trial exponential wave-function, so the two values cannot be compared.

**Find the minimum  $\langle H \rangle$ :**

$$\begin{aligned} \langle H \rangle_{\min} &= \left[ \frac{\hbar^2}{m} \left( \frac{6}{5} \right) b^2 - \frac{3b}{2} \alpha \right]_{b=5m\alpha/8\hbar^2} = \frac{\hbar^2}{m} \left( \frac{6}{5} \right) \frac{5^2 m^2 \alpha^2}{8^2 \hbar^4} - \frac{15m\alpha^2}{16\hbar^2} \\ &= \frac{15m\alpha^2}{32\hbar^2} - \frac{15m\alpha^2}{16\hbar^2} = -\frac{15}{32} \frac{m\alpha^2}{\hbar^2}. \end{aligned} \quad (2.4)$$

This is negative, and slightly higher (less negative) than the exact value. It is off the true value by only 6%. Thus we see that a qualitatively “good” trial wave-function provides much better results than a qualitatively “bad” one.

### 2.1.3 The Exact Solution

We now find the exact solution by solving the Schrodinger eigenfunction/eigenvalue equation. In a classically forbidden region where the potential is constant,  $\psi$  is always a decaying exponential:

$$-\frac{\hbar^2}{2m} \psi'' + V\psi = E\psi, \quad \psi'' - \frac{2m}{\hbar^2}(V - E)\psi = 0. \quad (2.5)$$

The solution is elementary:

$$\psi(x) = Ae^{\pm bx} \quad \text{where} \quad b = \frac{\sqrt{2m(V - E)}}{\hbar}. \quad (2.6)$$

$E$  is the as-yet unknown eigenvalue for energy, and  $b$  is just a notational convenience to simplify the formulas. (This  $\psi$  is the form we guessed for one of our variational wave-functions, so that variational  $\langle H \rangle$  must be exact.)

As always, to find the allowed (i.e., quantized)  $E$ , we must apply the auxiliary conditions. These are often boundary conditions, but with the delta-function potential, they are the matching conditions at the  $\delta$ -function. The first condition is that  $\psi$  must be continuous at  $x = 0$ :

$$\psi(0^-) = \psi(0^+),$$

which simply means (in this case) that  $A$  for the left half of  $\psi$  must equal  $A$  for the right half. The second matching condition is that  $\psi'$  must satisfy the SE at  $x = 0$ . This means the discontinuity in  $\psi'$  must be “just right.” We quantify this by integrating (2.5) across the discontinuity:

$$\psi'(0^+) - \psi'(0^-) = \int_{0^-}^{0^+} \psi''(x) dx = \frac{2m}{\hbar^2} \int_{0^-}^{0^+} (V - E)\psi(x) dx.$$

Note that both sides scale with the normalization  $A$ , which means we can ignore  $A$  here. On the LHS we get:

$$\psi'(x) = \pm be^{\pm bx} \quad \Rightarrow \quad \psi'(0^+) - \psi'(0^-) = (-be^0 - be^0) = -2b.$$

In the RHS, only the  $V$  term contributes:

$$\frac{2m}{\hbar^2} \int_{0^-}^{0^+} V(x)\psi(x) dx = \frac{2m}{\hbar^2} \int_{0^-}^{0^+} -\alpha\delta(x)e^{bx} dx = -\frac{2m\alpha}{\hbar^2}.$$

Equating the LHS and RHS:

$$-2b = -\frac{2m\alpha}{\hbar^2} \quad \Rightarrow \quad b = \frac{m\alpha}{\hbar^2}.$$

This is the energy eigenvalue equation, written in terms of  $b$ . (Because we guessed the exact form of  $\psi$  above, this eigenvalue equation is identical to the minimum energy equation, (2.1).) To find the eigenvalue  $E$ , we simply substitute (2.6) for  $b$ , and use  $V = 0$  outside the discontinuity:

$$\frac{\sqrt{2m(V - E)}}{\hbar} = \frac{m\alpha}{\hbar^2}, \quad 2m(0 - E) = \frac{m^2\alpha^2}{\hbar^2}, \quad E = -\frac{m\alpha^2}{2\hbar^2}.$$

This agrees with our lucky-guess variational estimate (2.2).

### 2.1.4 A Smooth Variation

I was challenged by a colleague to try a gaussian as a trial wave-function. Gaussians are generally “good” functions, and usually integrable in closed form. The question is: will a gaussian perform poorly like a cosine, because the gaussian is also smooth and flat in the middle, or will a gaussian perform well, like the kinky trials, because it’s, well, a gaussian. So here we go:

**Normalize:**  $\psi(x) = A \exp(-bx^2)$ . The gaussian integral is well known,  $\int_{-\infty}^{\infty} \exp(-ax^2) dx = \sqrt{\frac{\pi}{a}}$ :

$$1 = \int_{-\infty}^{\infty} \psi^*(x)\psi(x) dx = A^2 \int_{-\infty}^{\infty} \exp(-2bx^2) dx = A^2 \sqrt{\frac{\pi}{2b}}, \quad A^2 = \sqrt{\frac{2b}{\pi}}.$$

$b$  has units of  $m^{-2}$ .

**Compute  $\langle H \rangle = \langle T \rangle + \langle V \rangle$ :** The average potential energy is straightforward from the definition of the delta-function, and its integral:

$$\langle V \rangle = \int_{-\infty}^{\infty} \psi^*(x)V(x)\psi(x) dx = -\alpha A^2 \int_{-\infty}^{\infty} e^{-bx^2} \delta(x)e^{-bx^2} dx = -\alpha \sqrt{\frac{2b}{\pi}}.$$

The kinetic energy is straightforward, since all the derivatives are well-defined:

$$\begin{aligned} \psi' &= -2bx \exp(-bx^2) \\ \psi'' &= -2b \left( x \exp(-bx^2) (-2bx) + \exp(-bx^2) \right) = -2b \left[ -2bx^2 \exp(-bx^2) + \exp(-bx^2) \right] \\ \langle T \rangle &= \int_{-\infty}^{\infty} \psi^*(x) \left( \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right) \psi(x) dx \\ &= \frac{-\hbar^2}{2m} A^2 (-2b) \left[ \int_{-\infty}^{\infty} e^{-bx^2} (-2b)x^2 e^{-bx^2} dx + \int_{-\infty}^{\infty} e^{-bx^2} e^{-bx^2} dx \right] \\ &= \frac{\hbar^2 b}{m} \sqrt{\frac{2b}{\pi}} \left[ (-2b) \int_{-\infty}^{\infty} x^2 e^{-2bx^2} dx + \int_{-\infty}^{\infty} e^{-2bx^2} dx \right]. \end{aligned}$$

We use another standard integral,  $\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{a^3}}$ :

$$\langle T \rangle = \frac{\hbar^2 b}{m} \sqrt{\frac{2b}{\pi}} \left[ (-2b) \frac{1}{2} \sqrt{\frac{\pi}{8b^3}} + \sqrt{\frac{\pi}{2b}} \right] = \frac{\hbar^2}{m} \sqrt{2} \left[ -b \sqrt{\frac{1}{8}} + b \sqrt{\frac{1}{2}} \right] = \frac{\hbar^2}{2m} b.$$

This is positive, as it must be.

**Find the  $b$  that minimizes  $\langle H \rangle$ :**

$$\frac{d\langle H \rangle}{db} = 0 = \frac{\hbar^2}{2m} - \alpha \sqrt{\frac{1}{2\pi}} b^{-1/2}, \quad b^{-1/2} = \frac{\hbar^2}{2m\alpha} \sqrt{2\pi}, \quad b = \frac{2m^2 \alpha^2}{\pi \hbar^4}.$$

**Find the minimum  $\langle H \rangle$ :**

$$\langle H \rangle_{\min} = \frac{\hbar^2}{2m} \left( \frac{2m^2 \alpha^2}{\pi \hbar^4} \right) - \alpha \sqrt{\frac{2}{\pi}} \sqrt{\frac{2}{\pi}} \frac{m\alpha}{\hbar^2} = \frac{m\alpha^2}{\pi \hbar^2} - \frac{2m\alpha^2}{\pi \hbar^2} = -\frac{m\alpha^2}{\pi \hbar^2}.$$

This is off by more than a factor of 1.5 (57% error), and is substantially worse than the  $(bx + 1)^{-2}$  trial wave-function, which is only 6% off. We conclude that kinky trial wave-functions are better estimators for this potential, which requires a kink in its exact solution.

### 2.1.5 A Less Steep Rational Function

We tried a rational function that was squared in the denominator. How well would a less-steep trial wave-function do? The graph is again qualitatively the same as before (Figure 2.1b):

$$\psi(x) = A \frac{1}{b|x|+1}, \quad b > 0.$$

Our commentary is more terse this time. ‘ $b$ ’ has units of  $m^{-1}$ . We now follow the standard four steps for the variational estimate of the ground state energy.

**Normalize:** We use the even-ness of  $\psi$ , so we double the right-half integral:

$$1 = 2A^2 \int_0^\infty (bx+1)^{-2} dx = -\frac{2A^2}{b} \left[ (bx+1)^{-1} \right]_0^\infty = \frac{2A^2}{b}, \quad A^2 = \frac{b}{2}.$$

**Compute  $\langle H \rangle = \langle T \rangle + \langle V \rangle$ :** The average potential energy is straightforward from the definition of the delta-function, and its integral:

$$\langle V \rangle = \int_{-\infty}^\infty \psi^*(x) V(x) \psi(x) dx = -\alpha A^2 \int_{-\infty}^\infty (1) \delta(x) (1) dx = -\frac{b}{2} \alpha, \quad \text{using } A^2 = \frac{b}{2}.$$

The kinetic energy is again a bit subtle, because we must take the 2<sup>nd</sup> derivative of  $\psi$ , which is infinite at  $x = 0$  (the kink), so we split up the integral:

$$\langle T \rangle = \int_{-\infty}^\infty \psi^*(x) \left( \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right) \psi(x) dx = \frac{-\hbar^2}{2m} \left[ \int_{0^-}^{0^+} \psi^*(x) \psi''(x) dx + 2 \int_{0^+}^\infty \psi^*(x) \psi''(x) dx \right].$$

The derivatives of  $\psi$  are:

$$\begin{aligned} \psi'(x > 0) &= Ab(-1)(bx+1)^{-2}, & \psi'(x < 0) &= Ab(+1)(-bx+1)^{-2} \\ \psi''(x) &= Ab^2(+2)(b|x|+1)^{-3} \end{aligned}$$

As before, we use the Fundamental Theorem of Calculus for the first integral:

$$\begin{aligned} \langle T \rangle &= \frac{-\hbar^2}{2m} \left[ \int_{0^-}^{0^+} \psi^*(x) \frac{\partial^2}{\partial x^2} \psi(x) dx + 2A^2 b^2 \int_{0^+}^\infty (bx+1)^{-1} 2(bx+1)^{-3} dx \right] && \text{Use } 2A^2 = b \\ &= \frac{-\hbar^2}{2m} \left\{ \psi^*(0) (\psi'(0^+) - \psi'(0^-)) + \frac{2b^3}{(-3)b} \left[ (bx+1)^{-3} \right]_0^\infty \right\} && (2.7) \\ &= \frac{-\hbar^2}{2m} \left\{ A^2 b(-1-1) - \frac{2}{3} b^2 (-1) \right\} \\ &= \frac{-\hbar^2}{2m} \left\{ -b^2 + \frac{2}{3} b^2 \right\} = \frac{+\hbar^2}{6m} b^2. \end{aligned}$$

As before, we can visualize the wave-function to be as in Figure 2.1c, in the limit as the near-delta-function goes to zero width. Note that in this transition region, the KE is positive ( $\psi$  is concave toward the axis). Also,  $\langle T \rangle$  is positive, as it must be on physical grounds. This holds true even though the kinetic energy is negative for all of the wave-function, except at the discontinuity (where kinetic energy is infinite for an infinitesimal distance).

Finally:

$$\langle H \rangle = \frac{\hbar^2}{6m} b^2 - \frac{b}{2} \alpha.$$

**Find the  $b$  that minimizes  $\langle H \rangle$ :**

$$\frac{d\langle H \rangle}{db} = 0 = \frac{\hbar^2}{3m} b - \frac{\alpha}{2}, \quad b = \frac{3m\alpha}{2\hbar^2}.$$

**Find the minimum  $\langle H \rangle$ :**

$$\begin{aligned} \langle H \rangle_{\min} &= \left[ \frac{\hbar^2}{6m} b^2 - \frac{b}{2} \alpha \right]_{b=3m\alpha/2\hbar^2} = \frac{\hbar^2}{6m} \frac{9m^2\alpha^2}{4\hbar^4} - \frac{3m\alpha^2}{4\hbar^2} \\ &= \frac{3m\alpha^2}{8\hbar^2} - \frac{3m\alpha^2}{4\hbar^2} = -\frac{3}{8} \frac{m\alpha^2}{\hbar^2}. \end{aligned} \quad (2.8)$$

This is higher (less negative) than the exact value, by 25%. Since the exact wave-function is an exponential, which decays steeper than any rational function, we find that this less-steep function is not as good as the steeper rational function. It is still far better than the gaussian.

### 2.1.6 Triangles on Trial

How about a simple triangle function? It has a kink, but decays in a funny way:

$$\psi(x) = A \left( 1 - \frac{|x|}{b} \right), \quad b > 0, \quad -b < x < b. \quad \psi(x) = 0 \text{ elsewhere.}$$

' $b$ ' has units of m. We now follow the standard four steps for the variational estimate of the ground state energy.

**Normalize:** We use the even-ness of  $\psi$ , so we double the right-half integral:

$$1 = 2A^2 \int_0^b (1-x/b)^2 dx = -\frac{2A^2 b}{3} \left[ (1-x/b)^3 \right]_0^b = \frac{2A^2 b}{3}, \quad A^2 = \frac{3}{2b}.$$

**Compute  $\langle H \rangle = \langle T \rangle + \langle V \rangle$ :** The average potential energy is straightforward from the definition of the delta-function, and its integral:

$$\langle V \rangle = \int_{-\infty}^{\infty} \psi^*(x) V(x) \psi(x) dx = -\alpha A^2 \int_{-\infty}^{\infty} (1) \delta(x) (1) dx = -\frac{3}{2b} \alpha, \quad \text{using } A^2 = \frac{3}{2b}.$$

The kinetic energy now has 3 kinks in it, at  $-b$ ,  $0$ , and  $+b$ . The local kinetic energy at a kink is proportional to the change in slope, but the proportion is weighted  $\psi$  at the kink. We find the average KE in the usual way, by integrating. We split the integral into its three non-zero steps:

$$\begin{aligned} \langle T \rangle &= \int_{-\infty}^{\infty} \psi^*(x) \left( \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right) \psi(x) dx \\ &= \frac{-\hbar^2}{2m} \left[ \int_{-b^-}^{-b^+} \psi^*(x) \psi''(x) dx + \int_{0^-}^{0^+} \psi^*(x) \psi''(x) dx + \int_{b^-}^{b^+} \psi^*(x) \psi''(x) dx \right]. \end{aligned}$$

The first and 3<sup>rd</sup> terms are zero, because  $\psi^*(-b) = \psi^*(b) = 0$ . Only the middle term contributes. The derivatives of  $\psi$  are:

$$\psi'(x > 0) = \frac{A}{b}, \quad \psi'(x < 0) = -\frac{A}{b}$$

As before, we use the Fundamental Theorem of Calculus for the integral, noting that  $\psi^*(0) = A$ :

$$\langle T \rangle = \frac{-\hbar^2}{2m} \psi^*(0) \left( \frac{-A}{b} - \frac{A}{b} \right) = \frac{-\hbar^2}{2m} A^2 \left( \frac{-1}{b} - \frac{1}{b} \right) = \frac{3\hbar^2}{2b^2 m}$$

Finally:

$$\langle H \rangle = \frac{3\hbar^2}{2b^2 m} - \frac{3}{2b} \alpha.$$

**Find the  $b$  that minimizes  $\langle H \rangle$ :**

$$\frac{\partial \langle H \rangle}{\partial b} = 0 = -\frac{3\hbar^2}{m} b^{-3} + \frac{3}{2} b^{-2} \alpha, \quad \frac{3\hbar^2}{m} = \frac{3}{2} b \alpha, \quad b = \frac{2\hbar^2}{m \alpha}.$$

Confirm that  $b$  has units of meters.

**Find the minimum  $\langle H \rangle$ :**

$$\langle H \rangle_{\min} = \frac{3\hbar^2}{2m} \frac{m^2 \alpha^2}{4\hbar^4} - \frac{3\alpha}{2} \frac{m \alpha}{2\hbar^2} = \frac{3m\alpha^2}{8\hbar^2} - \frac{3m\alpha^2}{4\hbar^2} = -\frac{3m\alpha^2}{8\hbar^2}.$$

This is slightly higher (less negative) than the exact value, by 25%. This agrees with the less-steep rational function, but was a little easier to compute. It is still far better than the gaussian.

### 2.1.7 How Steep Is Steep?

We've seen that the steeper rational function performed better than the less-steep one. We must certainly ask, then, what is the best steepness? To find out, we create a *two*-parameter trial wave-function, with steepness as one parameter:

$$\psi(x) = A(b|x|+1)^{-n}, \quad b, n > 0.$$

It satisfies the requirement for a kink at  $x = 0$ , and is normalizable (if  $n > 1/2$ ).  $b$  has units of  $m^{-1}$ . We now follow the standard four steps for the variational estimate of the ground state energy. The algebra is a little more involved than before, but straightforward.

**Normalize:** We use the even-ness of  $\psi$ , so we double the right-half integral:

$$1 = 2A^2 \int_0^\infty (bx+1)^{-2n} dx = \frac{2A^2}{(-2n+1)b} \left[ (bx+1)^{-2n+1} \right]_0^\infty = \frac{2A^2}{(2n-1)b}, \quad A^2 = \frac{(2n-1)b}{2}.$$

We see this agrees with our previous calculations for  $n = 1$  and  $n = 2$ .

**Compute  $\langle H \rangle = \langle T \rangle + \langle V \rangle$ :** The average potential energy is straightforward from the definition of the delta-function, and its integral:

$$\langle V \rangle = \int_{-\infty}^\infty \psi^*(x) V(x) \psi(x) dx = -\alpha A^2 \int_{-\infty}^\infty (1) \delta(x) (1) dx = -\frac{(2n-1)b}{2} \alpha, \quad \text{using } A^2 = \frac{(2n-1)b}{2}.$$

The kinetic energy is a little bit subtle, because we must take the 2<sup>nd</sup> derivative of  $\psi$ , which is infinite at  $x = 0$  (the kink), so we split up the integral:

$$\langle T \rangle = \int_{-\infty}^\infty \psi^*(x) \left( \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right) \psi(x) dx = \frac{-\hbar^2}{2m} \left[ \int_{0^-}^{0^+} \psi^*(x) \psi''(x) dx + 2 \int_0^\infty \psi^*(x) \psi''(x) dx \right].$$

The derivatives of  $\psi$  are:

$$\begin{aligned} \psi'(x > 0) &= Ab(-n)(bx+1)^{-n-1}, & \psi'(x < 0) &= Ab(+n)(-bx+1)^{-n-1} \\ \psi''(x) &= Ab^2 n(n+1)(b|x|+1)^{-n-2} \end{aligned}$$

As before, we use the Fundamental Theorem of Calculus for the first integral:

$$\begin{aligned}
 \langle T \rangle &= \frac{-\hbar^2}{2m} \left[ \int_0^{0^+} \psi^*(x) \frac{\partial^2}{\partial x^2} \psi(x) dx + 2A^2 b^2 \int_0^\infty (bx+1)^{-n} n(n+1)(bx+1)^{-n-2} dx \right] \\
 &= \frac{-\hbar^2}{2m} \left\{ \psi^*(0) (\psi'(0^+) - \psi'(0^-)) + \frac{(2n-1)n(n+1)b^2}{(-2n-1)} \left[ (bx+1)^{-5} \right]_0^\infty \right\} \quad \text{Using } 2A^2 = (2n-1)b \\
 &= \frac{-\hbar^2}{2m} \left\{ A^2 b(-n-n) - \frac{(2n-1)n(n+1)}{2n+1} b^2 (-1) \right\} \\
 &= \frac{-\hbar^2}{2m} \left\{ -\frac{2n-1}{2} (2n)b^2 + \frac{(2n-1)n(n+1)}{2n+1} b^2 \right\} = \frac{-\hbar^2}{2m} \left[ -(2n-1)n + \frac{(2n-1)n(n+1)}{2n+1} \right] b^2 \\
 &= \frac{-\hbar^2}{2m} (2n-1) \left[ -\frac{(2n+1)n}{2n+1} + \frac{n(n+1)}{2n+1} \right] b^2 \\
 &= \frac{+\hbar^2}{2m} \frac{(2n-1)n^2}{2n+1} b^2.
 \end{aligned}$$

Again, this agrees with our previous calculations for  $n = 1$  and  $2$ , (2.7) and (2.3). Also,  $\langle T \rangle$  is positive, as it must be on physical grounds.

Finally:

$$\langle H \rangle = \frac{\hbar^2}{2m} \frac{(2n-1)n^2}{2n+1} b^2 - \frac{(2n-1)b}{2} \alpha.$$

**Find the  $b$  that minimizes  $\langle H \rangle$ :**

$$0 = \frac{d\langle H \rangle}{db} = \frac{\hbar^2}{m} \frac{(2n-1)n^2}{2n+1} b - \frac{(2n-1)\alpha}{2}, \quad b = \frac{2n+1}{2n^2} \cdot \frac{m\alpha}{\hbar^2}.$$

This agrees with past calculations. Since we have two parameters, we must also find the  $n$  that minimizes  $\langle H \rangle$ . Looking at the expression above for  $\langle H \rangle$ , we see that will be complicated. However, if we wait until we find  $\langle H \rangle_{\min}$  over  $b$  for a given  $n$ , then  $b$  is eliminated from the formula. We can then minimize w.r.t  $n$  more easily.

**Find the minimum  $\langle H \rangle$  over  $b$ :**

$$\begin{aligned}
 \langle H \rangle_{\min} &= \frac{\hbar^2}{2m} \left( \frac{(2n-1)n^2}{2n+1} \right) \frac{(2n+1)^2}{4n^4} \frac{m^2 \alpha^2}{\hbar^4} - \frac{(2n-1)(2n+1)m\alpha^2}{4n^2 \hbar^2} \\
 &= \frac{1}{2} \frac{(2n-1)(2n+1)}{4n^2 \hbar^2} m\alpha^2 - \frac{(2n-1)(2n+1)}{4n^2 \hbar^2} m\alpha^2 \\
 &= -\frac{(2n-1)(2n+1)}{8n^2 \hbar^2} m\alpha^2.
 \end{aligned}$$

We confirm again agreement with past calculations (2.8) and (2.4).

We can now minimize w.r.t  $n$ . The only factor that varies is the fraction above. Maximizing the fraction minimizes the energy (because of the leading negative sign). Therefore, maximize:

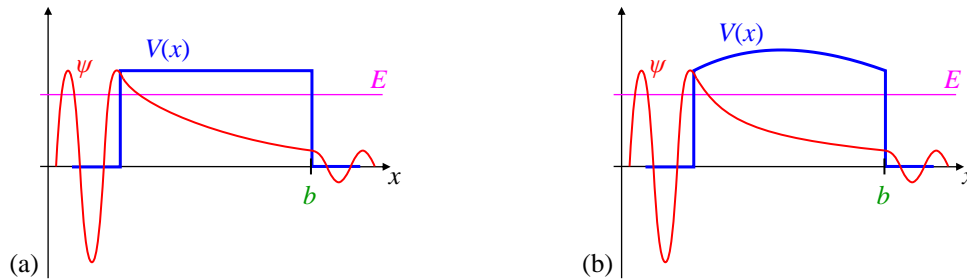
$$\frac{(2n-1)(2n+1)}{8n^2} = \frac{4n^2 - 1}{8n^2} = \frac{1}{2} - \frac{1}{8n^2}.$$



Larger  $n$  is better, and the limit of this fraction is  $1/2$ . This limit is the exact value for the ground state energy. You might be tempted to think this is qualitatively consistent with the fact that an exponential drops off faster than any power, so the larger the power (faster dropoff) the better. However, the faster exponential dropoff occurs for large  $x$ ; in our variational application, all the action is focused around the origin. I think it is probably coincidence that we recover the exact value in the large  $n$  limit.

## 2.2 The WKB Approximation

### 2.2.1 WKB For Tunneling



**Figure 2.2** (a) A barrier of constant potential. (b) A barrier of varying potential.

WKB can be used for another application: tunneling (Figure 2.2). In this case, the particle is unbound, so there is no quantization or quantization condition. In contrast to the bound case, where we were focused on the phase of  $\psi$ , but not its amplitude, for tunneling we are concerned only with its amplitude.  $\psi$  doesn't oscillate, so it has no phase.

We approximate the tunneling probability by adding up the neperes of attenuation through the barrier. Recall that, for a constant potential  $V$  (Figure 2.2a), the wave-function  $\psi$  penetrates into a classically forbidden region according to:

$$\psi_{incident} = e^{ikx}, \quad E = \frac{(\hbar k)^2}{2m}$$

$$\psi(x) = ce^{+\kappa x} + de^{-\kappa x} \quad \text{where} \quad \kappa \equiv \frac{\sqrt{2m(V-E)}}{\hbar}$$

When the probability of tunneling is small, it can be shown that  $c$  is small, and we ignore that term. Then the probability of tunneling is approximately:

$$\text{Pr}(tunneling) = |de^{-\kappa(b-a)}|^2 \equiv |de^{-\gamma}|^2 = |d|^2 e^{-2\gamma} \quad \text{where} \quad \gamma \equiv \kappa(b-a) \text{ in neperes}$$

$\gamma$  is the number of neperes of attenuation, sometimes called the number of “e-foldings.”  $\kappa$  has units of neperes/meter. Recall that two successive attenuation processes have a total neperes of attenuation that is the sum of the neperes for the two processes:

$$e^{-\gamma_1}e^{-\gamma_2} = e^{-(\gamma_1+\gamma_2)}$$

Also note that the unit of neper, like the radian, is equivalent to dimensionless. The matching coefficient  $d$  is typically of order 1, and therefore neglected.

More generally,  $V(x)$  is a function of position (Figure 2.2b), so  $\kappa(x)$  is also a function of position. Similar to the bound potential case, if  $(V-E)$  varies slowly compared to  $1/\kappa$ , we can approximate  $\kappa(x)$  from its local value. The total attenuation factor over some interval is then:

$$e^{-\gamma} = \exp\left(-\int_a^b \kappa(x) dx\right) \quad \text{where} \quad \kappa(x) \equiv \frac{\sqrt{2m(V(x)-E)}}{\hbar}$$

[Gri p324] gives an estimate for a nuclear lifetime based on a model of an alpha particle bouncing back and forth between the “walls” of the nuclear binding force. Presumably, at each “collision” with the wall, there is some chance the alpha particle tunnels outside the nucleus, and is ejected into the environment. This model is known to be flawed, even though it gives a reasonable result in this particular case. In other cases, it gives wildly inaccurate predictions. One limitation of this model is that it ignore quantum resonances between the particle wavelength and the size of the nuclear binding “box.”

## 3 Desultory Topics in Quantum Mechanics

### 3.1 Unobserved Things Affect Observed Things

This is the essence of “quantum weirdness.” The unobserved wave-function oscillates and interferes with itself. The interference leads to *observable* consequences: an interference pattern. However, the oscillation and interference itself cannot be observed. If we observe the particle, its wave-function collapses, and alters the time evolution compared to the unobserved wave. In other words, we need QM to compute the *unobserved* time evolution, so that we can eventually compute the *observed* consequences.

## 4 Quantum Electromagnetic Radiation

### 4.1 Quantum “Fluctuations” and Electromagnetic Zero Point Energy

#### Abstract

Contrary to common claims, quantum physics has no phenomena of fluctuations or any resulting particle field zero-point energy (ZPE or vacuum fields). While particles in a simple harmonic oscillator have a measurable ground-state energy, QFT fields are fundamentally different, and have no such thing. There is no reason to suspect vacuum fields exist from even basic theory, so the burden of proof lies with claims of their existence, not with their non-existence in QFT. There is no experimental result that requires more than standard (non-fluctuating) quantum theory for its analysis and understanding; this includes the phenomena of spontaneous emission, the Lamb shift, and the Casimir force. We examine many of the claims of observable consequences of vacuum fields, and discuss why they are not valid. For vacuum fields to be real, there must be some larger theory that fully reproduces all results of QED, and *additionally* includes observable consequences of vacuum fields.

#### 4.1.1 Introduction

There are no fluctuations in quantum theory. This is not an opinion; it is fact. Just look at the equations. The time-evolution of *all* quantum states is given by the Schrodinger Equation (SE):

$$\frac{\partial}{\partial t}|\psi\rangle = \frac{-i}{\hbar}\hat{H}|\psi\rangle.$$

Evolution is smooth, and continuous, and (in realistic systems) asymptotically approaches either stationary states, or particles far separated and no longer interacting. (In idealized situations, the time-evolution can be smooth, continuous, and *periodic*.) As shown by the SE above, quantum time evolution is never sporadic, discontinuous, or fluctuating (not even when taking a measurement).

There are no fluctuations in quantum theory.

Therefore, any analysis that includes phrases like, “Consider the fluctuations of a quantum system ...” is not rigorous. *Sometimes*, averaging over alleged “fluctuations” gives a right answer, because it leads to a mathematical average that is the same as averaging over the wave-function.

Any discussion of quantum fluctuations is *outside* quantum physics. However, there is substantial evidence *against* fluctuations, and in support of quantum theory. There is no experimental result that requires more than standard (non-fluctuating) quantum theory for its analysis and understanding [Jaffe 2005 p4]. This includes the phenomena of spontaneous emission, the Lamb shift, and the Casimir force, which many references incorrectly cite as examples of vacuum fields. In this section [of a chapter, not of a paper], we explore many of the common arguments over “quantum fluctuations,” focusing on the electromagnetic field.

This section assumes a background of basic hamiltonian physics and quantum mechanics. You do *not* need to be well-versed in QED, but some exposure helps. We proceed as follows:

- Definitions, and an unambiguous statement of the issue
- Simple proof that ZPE and fluctuations are unobservable
- The search for a quantum theory, such as QED
- Photon absorption
- Spontaneous emission
- Lamb shift
- Casimir force

#### 4.1.2 Definitions, and Statement of the Issue

There are several distinct ideas that are all sometimes referred to as “quantum fluctuations,” including:

- zero point energy (ZPE)
- non-zero variance of vacuum field states, taken as a dynamic randomly-varying classical-like field
- measurements producing sample distributions

- dark energy

The nonzero variance and sample distributions are also sometimes called “uncertainty.” We distinguish electromagnetic zero-point *energy* from vacuum *fluctuations*, though in the literature, they are often spoken of as one.

Our notation follows [Sc&Zu 1997], where the electric field and its operator look different:

$$\mathcal{E} \equiv \text{classical electric field}; \quad \hat{E} \equiv \text{quantum electric field operator}; \quad E \equiv \text{energy} .$$

We start with some phenomena that *are* observable: CMB temperature variations across the sky, matter ground-state energy (such as atoms in a lattice), thermal fluctuations, and dark energy.

**Matter ground-state-energy:** Ground state energy of *matter* oscillators is real, and measurable. For example, the atoms in a lattice have energy in their ground state, and it is measurable by melting the solid [ref Pathria??].

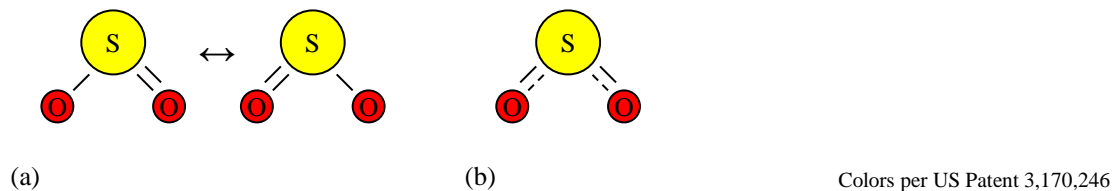
**Thermal fluctuations:** There are *thermal* fluctuations in both classical and quantum physics. They are truly dynamical, time-varying properties. Also, the Unruh effect says that in a space where inertial observers measure a vacuum, accelerated observers *do* measure a thermal bath of particles. However, thermal fluctuations do not concern us here.

**Dark energy:** Dark energy (aka the “cosmological constant”) is real. It is well supported by astrometry. Its origin is unknown. As of 2019, all theoretical attempts to explain dark energy by any field ZPE have failed. The measured dark energy density is  $\sim 10^{100}$  times smaller than the smallest vacuum energy one could reasonably get from any EM zero-point energy, if it existed [ref??]. It is *possible* that ZPE has some gravitational effects from its energy density, but there is no evidence to suggest it *does*. Dark energy exists in some form, even if it is only a term in the law of gravity.

Henceforth, we consider only non-gravitational effects,  
such as spontaneous emission, the Casimir force, the Lamb shift, etc.

**Measurements:** CMB temperature variations across the sky are believed to be the result of the inherent *uncertainty* of quantum measurements. The big bang produced a superposition of conditions of different temperatures. Measuring the sky temperature collapsed the quantum state to a specific temperature at each point in the sky. The temperatures are not fluctuating, and they never did: they’re the same now as they were yesterday, and will be tomorrow. They vary over the sky because taking a measurement produces a single sample of temperature from the population of temperatures defined by the quantum state.

One form of the claim of vacuum fluctuations starts with “quantum uncertainty.” But recall what that means: if I measure a set of identically prepared systems, I might get different results for each. “Quantum uncertainty” is a measure of the variation in such measurements. Quantum uncertainty is not about *fluctuations*; it’s about differences in measurements of identical systems. Do we say that an electron in an atom is “fluctuating” all around the nucleus? Is it zipping around at super-speeds from place to place? No, the electron state is stationary (constant in time), and nothing is “fluctuating.” Similarly, do we say the particle in a quantum simple harmonic oscillator is “fluctuating” around the equilibrium point? No, the particle is in a stationary state, and is in a superposition of *all* positions at *all* times. Do we say that SO<sub>2</sub> fluctuates between two resonance bond forms (Figure 4.1a)? No, the two S-O bonds are identical, and stationary. Thus, the uncertainty in the vacuum electric field simply means *if* we could measure the E-field of the vacuum, we would get different values for each measurement; uncertainty does *not* say that the E-field is *fluctuating*.



**Figure 4.1:** (a) Sulfur dioxide does not “resonate” between bond structures (two shown). (b) Empirically, the two S-O bonds are identical, with properties between single and double.

Uncertainty is not fluctuations.  
 QM consists of *distributed* particles (with wave functions), but not fluctuations.

## No Superheroes

Finally, we stress a precaution when relying on the great physicists of the past. During some of their lives, QED did not exist, QFT did not exist, and coherent states of the EM field were not understood until 1963. Quantum physics was still in its adolescence. It is no besmirchment of quantum pioneers to note that sometimes they were wrong. Many famous quotes have the physics wrong, and deification has hobbled physicists for years, or even decades. We know better now.

### 4.1.3 Simple Proof That EM ZPE is Unobservable

To preview the endpoint of our discussion, here is a simple proof that ZPE of EM fields cannot ever be observed (outside gravity, as noted above). This proof is rigorous, and should be unarguable. Recall the commonly cited electromagnetic hamiltonian, “derived” from the square of the electric field operator:

$$\hat{H}_{em} \propto \hat{a}^\dagger \hat{a} + \frac{1}{2}.$$

1. All systems are governed by a hamiltonian, and the hamiltonian representation of ZPE is an additive constant (at each frequency) of  $(1/2)\hbar\omega$ .
2. It is well known that an additive constant in the hamiltonian has no effect on the dynamics (i.e. equations of motion), hence the ZPE cannot ever be observed.

QED (pun intended).

Zero-point electromagnetic fields are unmeasurable in principle, and therefore do not exist. This fact is contained in the arguments of [Jaffe 2005], [Nikolic 2016], and [Grundler 2013], though (we think) somewhat obscured. Either vacuum fields interact with other things, or they don't. If they interact, there must be an interaction term in the hamiltonian. There is no such term, so there is no interaction, so they are not observable, and so they do not exist.

However, even with this proof, since the claim of observable EM ZPE is so widely disseminated, we continue with more thorough discussions. These include some common specific claims, and why they are not valid.

### The Hamiltonian and the “Origin” of Electromagnetic Vacuum Fields

QED is a hamiltonian theory, and the search for a hamiltonian theory is the search for *the* hamiltonian. No hamiltonian, no theory. Know hamiltonian, know theory. For example, how do we know that the classical EM interacting hamiltonian is  $(\mathbf{p} - e\mathbf{A}/c)^2/2m$ ? Because it agrees with experiment.

Let us search for the QED hamiltonian. Canonical quantization of the EM field starts by *assuming* the field is like a harmonic oscillator, and then combines *analogy* with a particle quantum SHO, and *classical* electromagnetic formulas, to *suggest* a form for the quantized field. (Note that canonical quantization itself cannot be derived: it is a *guess* based on an analogy.) Following this analogy gives a particular ordering for the quantum operators  $\hat{a}$  and  $\hat{a}^\dagger$  that results in a ZPE. Without going into detail:

$$H_{em,classical} \propto \mathcal{E}^2 + B^2 \quad \text{where } \mathcal{E}, B \equiv \text{classical fields} \quad \rightarrow \quad (4.1)$$

$$\hat{H}_{em} \propto \hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger = 2\hat{a}^\dagger \hat{a} + 1 \propto \hat{a}^\dagger \hat{a} + \frac{1}{2}.$$

Then when the prefactors are included:

$$\text{vacuum energy} = \langle 0 | \hat{H}_{em} | 0 \rangle = \frac{1}{2} \hbar \omega.$$

However, the *only* constraint that classical electromagnetics imposes on QED is that QED must reduce to the classical equations in the high-energy limit (and even then only for coherent photon states). QED cannot be *derived* from classical EM, and in particular, classical EM *cannot* determine the order of quantum operators.

The hamiltonian (4.1) above does reduce to classical EM as needed. On the other hand, changing the original term  $\hat{a}\hat{a}^\dagger$  to  $\hat{a}^\dagger\hat{a}$  yields an alternative hamiltonian, with no constant, that *also* satisfies the classical limit:

$$\hat{H}_{em} \propto \hat{a}^\dagger\hat{a}.$$

This produces identical physics to the guessed hamiltonian (4.1), because the constant has no effect: it is neither a source nor a sink of energy. It is always present, cannot be removed, and cannot interact with anything.

How can we decide on a hamiltonian? In a sense, we don't have to, because both produce identical physics. They are equivalent. In this light (yes, I said it), why would we choose the quantum electromagnetic hamiltonian to be  $\hat{H}_{em} \propto \hat{a}^\dagger\hat{a} + \frac{1}{2}$ , when all the results are obtained just as well with  $\hat{H}_{em} \propto \hat{a}^\dagger\hat{a}$ ? There is *no* a priori reason to choose the former, and abundant reason to discard it and choose the latter. Virtually all QFT treatments make exactly this choice. Again, the *classical* notion that the hamiltonian  $H_{em}$  is proportional to  $\mathcal{E}^2$  has no relevance to our choice between these two *quantum* hamiltonians (hamiltonia?).

Throughout the development of QFT (and other theories), physicists make choices of “appropriate” solutions, and discard “extraneous” (or “unphysical”) solutions, on the basis of physical plausibility. For example, we *choose* the Green function propagator for a particle in such a way that causality is preserved, and discard other “unphysical” solutions to the equation of motion. There is no more reason to believe the vacuum is filled with infinite energy density, than there is to believe it is filled with infinite “negative energy” electrons.

**Matter of fact:** The idea of the EM field ZPE is fundamentally different than the matter SHO ground-state energy. Canonical field quantization is done by *analogy* to quantized matter, but does not actually follow the same derivation, or *any* derivation: it is a guess. Whereas matter ground-state energy results from physically *constructing* a harmonic oscillator, EM field ZPE is taken as an inviolate constant of all EM fields, regardless of the physical system involved. That's a big difference. Virtually all QFT books briefly discuss field ZPE, and immediately discard it: [P&S 1995 p21-22] says the ZPE “... cannot be detected experimentally... . We will therefore ignore this infinite constant term in all of our calculations.”; [M&S 2010 p8m] says, “This is an infinite constant, which is of no physical significance.” It is never used in any subsequent calculation, so it clearly has no effect on the physics.

Thus, in the search for the QED hamiltonian, we have found it, and it is  $\hat{H}_{em} \propto \hat{a}^\dagger\hat{a}$ .

Zero-point energy or fluctuations are no more a part of QFT than the luminiferous ether is a part of electromagnetics.

**What is a hamiltonian?** Recall that the hamiltonian of a system is not always the system energy, and an energy function is not always the hamiltonian: in QFT, the Dirac hamiltonian is certainly not the energy of a fermion; and classically, a charged particle in an EM field has energy  $p_{kin}^2/2m + V(\mathbf{r})$ , but that is not the hamiltonian (it has no EM field in it!). As always, though, the hamiltonian *is* the generator of time evolution.

Classically, Hamilton's equations immediately show that an additive constant in the hamiltonian has no effect on the dynamics of any arbitrarily complex system:

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}.$$

Similarly, any *arbitrarily complex* quantum system has a hamiltonian, and the time evolution of *any* observable is uniquely determined by the commutator of the Heisenberg operators:

$$\frac{d\hat{O}}{dt} = \frac{i}{\hbar} [\hat{H}, \hat{O}] + \frac{\partial \hat{O}}{\partial t}.$$

A constant added into the system hamiltonian cannot have *any* observable consequence: a constant commutes with all operators, so contributes nothing to the time evolution of any observable. The so-called “vacuum energy” is a

constant (a sum of constants, one for each  $\omega$ ), and so cannot have *any* observable consequences. Not spontaneous emission, not the Lamb shift, not the Casimir force, not anything.

#### 4.1.4 Burden of Proof: Are Vacuum Fluctuations a Credible Claim in the First Place?

Below, we consider evidence *against* vacuum fields, but even before pursuing such evidence, let us consider how credible are the claims of “vacuum fluctuations” or “zero point energy” in the first place. In other words, where is the burden of proof best lain?

Given that the two hamiltonians under consideration yield *identical* physics in all cases (see proof above), any claim of observable vacuum fluctuations is beyond implausible. If such observations are possible, they must be described by physics *outside* QED. QED is the most accurate physical theory ever developed, with experimental agreement on the magnetic dipole moment of the electron to 14 digits. No other physical theory even comes close.

[Jaffe 2005] says, “Certainly there is no experimental evidence for the reality of zero-point energies in quantum field theory (without gravity). ... no known phenomenon, including the Casimir effect, demonstrates that zero-point energies are real.” [Gründler 2013] concurs: “In total, no experimental evidence at all is indicating the measurable, observable existence of the zero-point energy of elementary quantum fields.” We emphasize: “no known phenomenon” demonstrates that ZPE is real.

Since the claim of vacuum fluctuations is not well-founded in the first place, we conclude there is little need to refute it. Nonetheless, we now consider several pieces of evidence against it, and the veracity of some claims for their existence.

#### 4.1.5 That’s Intense: Photon Absorption

We compute here two simple, but important results: (1) the average of the squared electric field ( $\mathcal{E}^2$ ), and (2) the photon radiation intensity, as operationally defined: in other words, what we can actually *measure*. We drop here all the pre-factors and normalization to focus on the main ideas.

$\langle \mathcal{E}^2 \rangle$ : Classically,  $\mathcal{E}^2$  is proportional to the wave intensity. For comparison, what is  $\langle \mathcal{E}^2 \rangle$  for a quantum number state  $|N\rangle$ ? For a single mode (some  $(\mathbf{k}, \lambda)$  pair):

$$\begin{aligned} \hat{E} &\propto \hat{a}^\dagger + \hat{a} \quad \text{is the electric field operator;} \\ \hat{E}^2 &\propto \hat{a}^{\dagger 2} + \hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger + \hat{a}^2; \\ \langle \mathcal{E}^2 \rangle &= \langle N | \hat{E}^2 | N \rangle \propto \langle N | \hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger | N \rangle = N + N + 1 \propto N + \frac{1}{2}. \end{aligned} \tag{4.2}$$

The “1/2” is the infamous “zero point energy.” So we have a ZPE in this *theoretical calculation*, but can we measure it? Classically,  $\mathcal{E}^2 \propto$  intensity. But in quantum systems,  $\langle \mathcal{E}^2 \rangle$  is hard to measure directly, so how do we define quantum intensity? For that we consider the *operational* computation.

**Probability of photon absorption:** We now compute something which *is* directly measurable: the rate of absorption of photons from an EM field. For detection by the photoelectric effect, this is the photoelectron rate, or ionization rate. The rate is proportional to the probability of absorbing a photon, i.e. of transitioning from an EM field state of  $|N\rangle$  to  $|N-1\rangle$ :

$$\text{absorption rate} \propto \langle N-1 | \hat{a} | N \rangle = \frac{\langle N | \hat{a}^\dagger \hat{a} | N \rangle}{\langle N-1 |}$$

This holds even for the vacuum state,  $|N=0\rangle$ : there is *zero* absorption from the vacuum. This experimental fact is at odds with a supposedly fluctuating E-field:

There is no zero-point effect on the absorption rate;  
fluctuations or not, you can’t absorb a photon you don’t have.

This result is so important that quantum optics *defines* intensity in terms of photo-absorption rate, rather than in terms of  $\mathcal{E}^2$  [Sc&Zu 1997 4.2.17 p114t]. There is no zero-point effect on the intensity of a light beam.



Mathematically, for single photon absorption, the quantum intensity is:

$$\langle I \rangle \equiv K \langle (N-1) \hat{a} | N \rangle^2 = KN \quad \text{where } K \text{ is a constant proportional to } \hbar\omega.$$

To pave the way for multi-photon absorption, the intensity is often written in terms of the  $\hat{E}^{(+)}$  and  $\hat{E}^{(-)}$  operators, which are proportional to  $\hat{a}$  and  $\hat{a}^\dagger$ :

$$\langle I \rangle = \langle N | \hat{E}^{(-)} \hat{E}^{(+)} | N \rangle = K \langle N | \hat{a}^\dagger \hat{a} | N \rangle = KN.$$

We can therefore define the intensity operator as  $\hat{I} = \hat{E}^{(-)} \hat{E}^{(+)}$ . This is an *operational* definition (i.e., definition of *observables*), which can be directly measured, rather than an abstract (and unjustified) analogy with classical physics.

This definition makes intuitive sense: intensity is proportional to the energy of the electric field (just like classically), which is proportional to the number of photons in the EM field.

([Sc&Zu 1997 p11b] at first loosely call the operator  $\hat{E}^2$  the “intensity” operator; however, they later explicitly correct themselves [Sc&Zu 1997 4.2.17 p114t], consistent with our discussion here.)

#### 4.1.6 Spontaneous Emission

Vacuum fluctuations are occasionally invoked as the “stimulus” for spontaneous radiation. While this may be comforting, it is quantitatively wrong. The measured spontaneous emission rate is twice that predicted by vacuum fluctuations, but agrees exactly with standard QED. Therefore, we must conclude that spontaneous emission is *not* due to EM vacuum fluctuations.

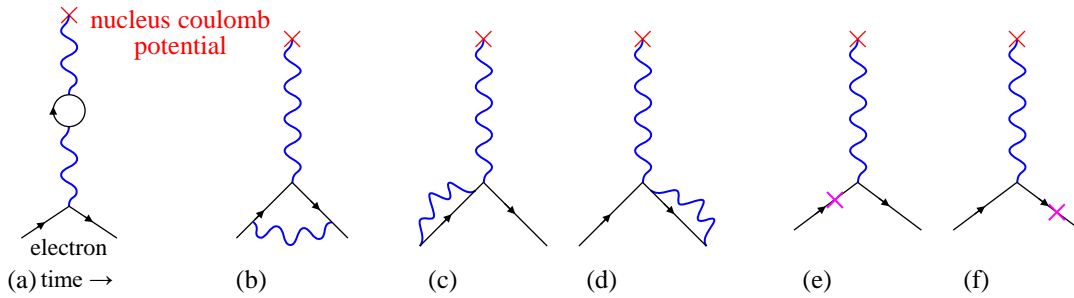
The discrepancy is simple to show: in brief, the vacuum would provide only half a quantum of stimulation,  $(1/2)\hbar\omega$ , but the spontaneous emission rate is proportional to a full quantum,  $\hbar\omega$ . In terms of photon number states:

$$R_{vac} \propto \langle 0 | \hat{E}^2 | 0 \rangle = K \left( \frac{1}{2} \right); \quad R_{QED} \propto \langle 1 | \hat{E}^2 | 0 \rangle = K \quad \text{where } K \equiv \text{constant}.$$

#### 4.1.7 The Lamb Shift

The Lamb shift is the measured energy difference between the  $2s_{1/2}$  and  $2p_{1/2}$  hydrogen levels. The real lesson of the Lamb shift is that quantum physics demands more than just a relativistic single-electron theory. It requires a fully relativistic quantum field theory, with particle creation and annihilation, including the quantized EM field. The Dirac and Klein-Gordon equations alone are insufficient. Historically, the Lamb shift was a major driver of the advancement of QED.

According to the single-particle Dirac equation, the  $j = 1/2$  hydrogen levels  $2s_{1/2}$  and  $2p_{1/2}$  are exactly degenerate (same energy). However, experiment revealed the  $2s_{1/2}$  level is higher by  $\Delta E/h = 1057.77 \pm 0.10$  MHz as of [Be&Sa 1957 p107t]. Using only QED, [Be&Sa 1957] accounts for the shift to  $\sim 0.6$  MHz. They make no reference at all to vacuum fluctuations or ZPE. Their derivation is quite involved, covering many sections of the book, but is essentially 2<sup>nd</sup>-order stationary-state perturbation theory. To get a feel for the complexity of QED, Figure 4.2 shows the 6 first-order Feynman diagrams that contribute to the Lamb shift, with the dominant contributions being the electron so-called “self-energy” diagrams (c to f) [M&S p187m]. None of them are related to vacuum fluctuations, because QED has no such concept. But the complexity of their derivation (and of modern QED) drives some people to reach for a simpler development.



**Figure 4.2** First order Lamb shift contributors are radiative corrections: (a) fermion loop; (b) vertex correction; (c) and (d) fermion “self-energy”; (e) and (f) fermion mass counter-term.

[Welton 1948] first suggested a visualization where the Lamb shift is estimated by appealing to fluctuations in the electron position, arising from fluctuations in the vacuum EM field [Sc&Zu p14-16]. [Welton 1948 p1157] calls this is “a semiquantitative calculation.” He notes, “we shall assume that the position fluctuation is a real concept, while we shall think of the energy of interaction ... as having no physical reality.” This is contradictory: the whole point of the Lamb shift is that the electron *energy is raised*. His summary [p1167] also says, “The result suffers, however, from the obvious disadvantage that a non-relativistic Hamiltonian was used ... where the non-relativistic assumption is clearly seriously in error.” Evidently, Welton did *not* think of this heuristic approach as proof of anything. We think it was an idea worthy of its time, put forth to see where it leads. In the end, it leads nowhere.

Welton’s view of the Lamb shift is based on hypothetical zero-point fluctuations, but not on zero-point energy (in contrast to the Casimir effect). The idea is that fluctuations in the vacuum EM field push the atomic electron around, and on average, increase its energy. Welton calls this “intuitive,” and claims the Lamb shift can be “estimated in a simple classical fashion.” He estimated that the energy change is proportional to the mean-squared electric field  $\langle \mathcal{E}^2 \rangle$  of the fluctuating vacuum. He claims his calculation gives rough agreement with the measured Lamb shift. Is this evidence of electric field vacuum fluctuations? No. We show below that the quantitative agreement is, in fact, weak.

In [Welton 1948], we see a common historical sequence: the original author of an idea (Welton) did *not* believe the effect was necessarily real, nor was his treatment any kind of “proof.” It is only later misunderstanding, rather than hard physics, that has grown into the current quantum legend.

Welton does arrive at the same indeterminate formula as Bethe for the energy shift, which contains an integral that is logarithmically divergent at both ends [Welton 1948 eq 9], [Bethe 1947 eq 11]. However, besides the limitations Welton noted, the fluctuation argument has several more flaws. For one, the divergent integral of the photon (or vacuum field) energy gives the Lamb shift the form:

$$\Delta E \propto \int_0^\infty \frac{d\omega}{\omega} = \ln \frac{\text{upper-cutoff}}{\text{lower-cutoff}}. \tag{4.3}$$

Therefore, *two* cut-offs must be *chosen* to give a finite result. Both Bethe and Welton agree that the upper cut-off should be of order  $mc^2/\hbar$ , where  $m \equiv$  electron mass. That may be a reasonable choice, but it can hardly be considered rigorous physics: different choices result in different “Lamb shifts.” Even [Bethe 1947] does not provide a good upper-cutoff, though later work does [ref??].

**Quantitative consequences of various cutoffs:** Let us consider some quantitative uncertainties in the calculation. [Bethe 1947] describes the argument of the logarithm as “very large.” In fact, the logarithm is fairly small, and so the cutoff choices have a large influence on the approximation. [Bethe 1947] used QED to accurately calculate the lower cut-off as 242 eV, which Bethe calls “an amazingly high value.” This yields an accurate Lamb shift of:

$$\Delta E \propto \ln \frac{mc^2}{242 \text{ eV}} = \ln \frac{511 \text{ keV}}{242 \text{ eV}} = 7.7 \rightarrow 1000 \text{ MHz}.$$

There is at least a factor of two uncertainty in either direction on the upper cutoff, and in fact, there could easily be a factor of  $2\pi$  uncertainty.  $\ln 2\pi \rightarrow \pm 1.8 = \pm 24\% = \pm 240 \text{ MHz}$  uncertainty in the estimate. Even if vacuum fluctuations reproduce this result, we already fall far short of *proof* of vacuum fluctuations.

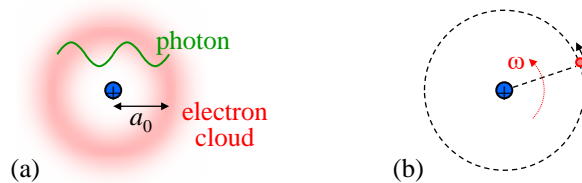
The lower cutoff is far more problematic. Vacuum fluctuations provide *no* quantitative justification for the lower cutoff. If we don't look at Bethe's QED result, and choose a lower cutoff based on only the information we have from the fluctuation argument, we get *much* worse results. For example, we could argue (loosely) that once a photon wavelength is longer than the orbital radius  $a_0$ , its field is roughly uniform across the atom, and therefore has no effect (Figure 4.3a). Therefore, the lower cutoff is of order  $a_0$ :

$$E_{lower} = hf = \frac{hc}{a_0} \approx 23 \text{ keV} \rightarrow \Delta E \propto \ln \frac{511 \text{ keV}}{23 \text{ keV}} = 3.1 \rightarrow 400 \text{ MHz}.$$

This is quite far from the measured value of ~1000 MHz, and again, there's probably another factor of 4 or so cutoff uncertainty to be combined into this result, making it even worse.

Trying again, perhaps the cutoff should be about the ionization frequency (13.6 eV), because ionization greatly changes the electron energy. The resulting Lamb shift would be:

$$\Delta E \propto \ln \frac{511 \text{ keV}}{13.6 \text{ eV}} = 10.5 \rightarrow 1370 \text{ MHz}.$$



**Figure 4.3:** (a) Lower wavelength photon cutoff? (b) Bohr model photon cutoff?

Ultimately, [Welton 1948] gives only an unphysical suggestion that the photon lower cutoff frequency could plausibly be of the order of the *classical* electron orbital frequency in a *Bohr orbit* of  $l = 1$ ! See Figure 4.3b [Sc&Zu 1997 1.3.7 p15m], [Welton p1165 “orbital frequency”]. Recall that the Bohr model is completely wrong, and in fact, an *s* state has an orbital frequency of zero ( $l = 0$ ). However, with that lower cutoff we get:

$$\hbar = I\omega = ma_0^2 \omega, \quad \omega = \frac{\hbar}{ma_0^2} \quad \Rightarrow \quad E_{lower} = \hbar\omega = \frac{\hbar^2}{ma_0^2} = 27 \text{ eV}$$

$$\Delta E \propto \ln \frac{511 \text{ keV}}{27 \text{ eV}} = 9.8 \rightarrow 1300 \text{ MHz}.$$

[Welton 1948] does not ever determine a reasonable lower cutoff, and none suggested above give good results. Instead, Welton uses Bethe's calculation of actual QED (242 eV), and claims that his classical electron Bohr orbit frequency of 27 eV makes Bethe's large cutoff “not surprising” [Welton 1948 p1165].

In sum, using only vacuum fluctuation reasoning, we could argue for any Lamb shift from about 300 - 1500 MHz. That's a wide range to take as “proof” of anything.

**Stability of Lamb shift:** If the Lamb shift were due to energy fluctuations of the buffeted electron, the measurements should show a wide distribution of energy shifts, depending on the value of the E-field fluctuation at the instant of measurement. In fact, the line shapes are well-measured [Be&Sa 1957 p105m], and such a distribution is not observed. The line shapes agree with QED predictions, with no fluctuations in sight. The stability of the Lamb shift is in fact evidence *against* vacuum fluctuations.

**Perpetual motion:** Dynamic random E-fields, as assumed in the vacuum fluctuation approach to the Lamb shift, would be a source of thermal energy to the electron. It could drive a perpetual motion heat engine (dumping into, say, the 3 K CMB). This is why Welton had to assume that the electric field forces and accelerations were real, but somehow the energy transfer was not. This is physically contradictory.

**Comments on the Lamb Shift**

**The meaning of “virtual photons:”** The Feynman diagrams of Figure 4.2 can be misleading if not carefully interpreted. The *wrong* interpretation is that photons (and other stuff) are constantly popping into and out of

existence: fluctuating in a quantum foam of violent activity. To see the proper interpretation, recall that the true eigenstates of a hydrogen-like atom are those of the full, interacting hamiltonian. A hydrogen  $2s_{1/2}$  state is a stationary state: nothing fluctuates or foams. However, we write this true state in the hamiltonian eigenbasis of non-interacting (free-field) theory, i.e. the momentum basis. The true eigenstate contains an infinite superposition of free-field components: some with zero photons, some with one photon, etc:

$$\begin{aligned}
 |\psi_{true}\rangle = & c_1 \underbrace{|\psi_{100}\rangle|vac\rangle}_{\substack{\text{Dirac state,} \\ \text{no photons}}} + c_2 \underbrace{|\psi_{100}\rangle|N_{k_1\lambda_1}=1\rangle}_{\substack{\text{Dirac state,} \\ \text{one photon}}} + c_3 |\psi_{100}\rangle|N_{k_1\lambda_2}=1\rangle + c_4 |\psi_{100}\rangle|N_{k_2\lambda_1}=1\rangle + \dots \\
 & + c_5 |\psi_{200}\rangle|vac\rangle + c_6 |\psi_{200}\rangle|N_{k_1\lambda_1}=1\rangle + \dots
 \end{aligned}$$

The Feynman diagrams in Figure 4.2 correspond to terms in this superposition.

The non-interacting basis is mathematically convenient, but physically meaningless. The hydrogen atom is *not* constantly radiating and absorbing photons (virtual or otherwise). Using the non-interacting basis (momentum basis) simply means that actual physical states must be written as superpositions of lots of momentum-basis states. The electrons no more radiate photons than they zip from place to place around the nucleus. The actual physical states are completely stationary, just like the particles in an SHO, and the chemical bonds of  $\text{SO}_2$ .

Recall, too, that being in a “superposition” is not a physically meaningful concept. Every state is a superposition in *some* basis, and an eigenstate in some others. However, being an eigenstate of the *hamiltonian* is physically meaningful: it is a stationary state.

**Summary of the Lamb shift:** Fluctuations are a visualization method that is sometimes thought to be helpful in picturing quantum effects. We believe that such imaginary fluctuations are misleading, and cause more harm than good. Rather than invoking vacuum fluctuations, 2<sup>nd</sup>-order stationary state perturbation theory yields the same expression for the major contributors to the Lamb shift, and uses proper quantum physics. Furthermore, QED provides a physical justification for reliable cutoffs, necessary for credible claims of experimental agreement.

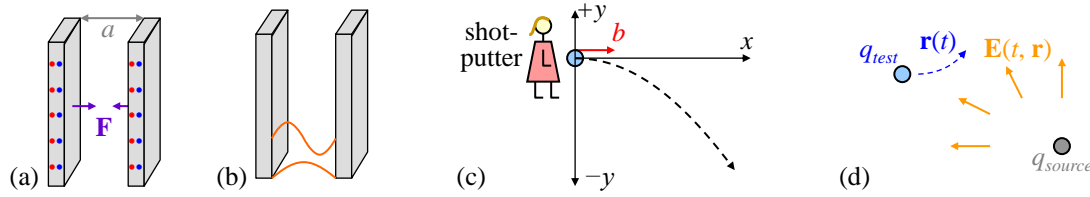
While the idea of the vacuum containing fluctuations of the E-field sometimes gives reasonable answers, it sometimes gives *incorrect* answers.

The Lamb shift contains no evidence in favor of vacuum fluctuations, and in fact, its stability is evidence against them.

#### 4.1.8 Casimir Force Is Not Due to Vacuum Energy

The Casimir force is the experimentally verified attraction between two neutral dielectric or conducting plates (Figure 4.4a). We now show that the Casimir force is *not* due to vacuum energy (e.g., [Nikolic 2016], [Jaffe 2005], [Grundler 2013]), despite frequent claims otherwise. We confine ourselves to the idealized case of perfectly conducting plates, where “the Casimir force is simply the (relativistic, retarded) van der Waals force between the metal plates” [Jaffe 2005]. Recall that van der Waals forces arise from the spontaneous polarization of charge-aggregates in each other’s presence (Figure 4.4a). For conductors, the polarization results from the nuclei and electrons pushing on each other, and inducing a dipole moment in the conductors. The direction of polarization (left or right) is randomly determined when the plates are brought near each other, by spontaneous symmetry breaking.

Casimir and Polder's original paper [Ca&Po 1948], which analyzes a related effect, does not refer to zero-point energy (ZPE) at all; instead, they compute direct EM potentials with a quantized EM field, and a simple dipole model of the atom. Their analysis is motivated by the retardation from the finite speed of light. Their use of QED automatically includes retardation, since QED is fully relativistic. Their calculation is essentially a QED computation of Van der Waals forces for polarizable bodies. (Note that [Ca&Po 1948] use the symbol  $\alpha$  for polarizability, not for the electromagnetic coupling.)



**Figure 4.4:** (a) The real cause of the Casimir force is spontaneous dipoles induced in the conductors. (b) Some classically allowed E-field modes. (c) Illustration of Hamiltonian mechanics. (d) System of test charge and electric field; the fixed source charge provides the external potential.

Only later did Casimir explore the force by considering a ZPE [Casimir 1999 p176b], [Casimir 1948]. The vacuum energy argument depends on hypothetical zero-point energy, but not fluctuations (in contrast to the Lamb shift). This argument (Figure 4.4b) claims that the conductors extinguish the vacuum EM field except at the classical frequencies allowed by the boundary condition of zero parallel E-field at the conductors. The extinction of the vacuum E-field is also said to extinguishes its zero-point energy (ZPE). Bringing the plates closer together allows fewer modes to exist, and therefore results in lower EM field energy. Since the system energy decreases with closer spacing, the work-energy theorem says there should be an attractive force between them: the lost EM energy does work on the plates. (Of course, the EM field energy is always infinite, so the calculation must subtract infinities to get finite energy differences, but this can be done definitively, and is not the error in the argument.)

The generally accepted formula from all references, including Casimir’s ZPE argument, is:

$$F = \hbar c \frac{\pi^2}{240a^4} \quad \text{where } a \equiv \text{plate separation} . \tag{4.4}$$

[Ma&Wo 1995 p509b] notes that the Casimir formula includes  $\hbar$  (which makes it look “quantum”), but not the fundamental electric charge  $e$ , which makes it look independent of charges, and thus due solely to fields.

Nonetheless, the argument is invalid, for multiple reasons. First, of course, is that it presumes an interaction between charges in the metal plates and the vacuum EM field. Such interactions are not in the interaction hamiltonian  $\hat{j}^\mu \hat{A}_\mu$ , and so are not part of QED.

**Impossibility of canceling the vacuum energy:** Another failure in the zero-point Casimir force development is that it states that the charges on the plates completely cancel the vacuum field. How is this accomplished? What quantum state of the EM field  $|\psi_{EM}\rangle$  do the plate charges excite, such that for most given frequencies  $\omega = c|\mathbf{k}|$ , the average of  $\mathcal{E}^2$  is zero? From (4.2) we must have:

$$\langle \psi_{EM} | \hat{E}^2 | \psi_{EM} \rangle = 0 \quad \text{where } \hat{E}^2 \propto \hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}) + \frac{1}{2} .$$

This implies:

$$0 = \langle \psi_{EM} | \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right) | \psi_{EM} \rangle = \underbrace{\langle \psi_{EM} | \hat{a}^\dagger \hat{a} | \psi_{EM} \rangle}_{\hat{E}^2} + \frac{1}{2} \Rightarrow \langle \psi_{EM} | \hat{a}^\dagger \hat{a} | \psi_{EM} \rangle = -\frac{1}{2} .$$

This is impossible (as is well-known), since  $\hat{a}^\dagger \hat{a}$  is a non-negative operator: this inner product is the magnitude squared of the ket  $\hat{a}|\psi_{EM}\rangle$ . This magnitude may be zero, but no less.

**Dependence on electromagnetic coupling:** [Jaffe 2005] shows that in fact, the simple formula (4.4) is actually the limiting result for large  $a \equiv e^2/\hbar c$ . Using a one-dimensional system for simplicity, the more precise formula *does* include the electric charge  $e$ :

$$F(a, e, m) = -\frac{e^2}{\pi} \int_m^\infty dt \frac{t^2}{\sqrt{t^2 - m^2}} \cdot \frac{\exp(-2at)}{4t^2 + 4et + e^2(1 - \exp(-2at))}, \quad \hbar = c = 1 \quad \text{[Jaffe 2005 eq. 8]}$$

where  $a \equiv$  conductor spacing;  $e \equiv$  electric charge;  $m \equiv$  field mass, later taken to 0.

Perfect conductors corresponds to taking  $e \rightarrow \infty$  ( $\alpha = 1/137$  is large in this case), and the dependence on  $e$  drops out. (Then taking  $m \rightarrow 0$  recovers the 1D force from a ZPE calculation.) Similarly, the 3D Casimir force *does* depend on  $e$ , and therefore on the interaction between fields and charges. The force is *not* a property of the vacuum alone.

**Improper hamiltonian:** Another failure of the vacuum-energy development is more subtle. We start with a simple example to help explain why. Our treatment is inspired by [Nikolic 2016]. Consider a well-known *classical* mechanics problem: an athlete throws a shot-put ball horizontally off a cliff (Figure 4.4c). All the usual idealizations apply: gravity is uniform, no air resistance, etc. Question: Use Hamiltonian mechanics to find its 2D motion, and explore the implications of an alternative “hamiltonian.”

Solution: We take the  $(x, y)$  origin as the launch point, and the horizontal velocity is constant, say  $b$ . The hamiltonian is (we take  $g > 0$ ):

$$H = T + V = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + mgy. \quad (4.5)$$

The hamiltonian of a system encodes *all* of its dynamics, and Hamilton’s equations give the *general* equations of motion. Note that this hamiltonian has no  $x$  dependence (only  $p_x$  dependence). Therefore, as expected, there is no  $x$ -force:  $F_x = -\partial H/\partial x = 0$ .

Given the auxiliary conditions in the problem (initial position and velocity), we solve the *general* equations of motion to get the *particular* solution (trajectory) of this problem. The motion as a function of time is well-known:

$$x = bt, \quad y = -\frac{1}{2}gt^2.$$

Let us now do simple rearranging to eliminate  $t$  in favor of  $x$  in the hamiltonian:

$$t = x/b \Rightarrow \quad y = -\frac{g}{2} \frac{x^2}{b^2}, \quad H_{alt} = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} - m \frac{g^2 x^2}{2b^2}.$$

At every point in the trajectory,  $H_{alt}$  is equal to the original hamiltonian (4.5) for this problem;  $H_{alt}$  is the total energy. However, this new “hamiltonian” has  $x$  dependence, and therefore an  $x$ -force:

$$F_x = -\frac{\partial H_{alt}}{\partial x} = m \frac{g^2}{b^2} x.$$

This means there is a horizontal force on the ball, which increases with distance!

So which is it? Both functions give the total energy, but which function is the correct hamiltonian? How can we tell?

The total energy is not always the hamiltonian, and the hamiltonian is not always the total energy.  
The *defining* property of the hamiltonian is that it is the generator of time evolution.

For example, in QFT, the Dirac hamiltonian is certainly not the energy, but it *is* the generator of time evolution. The important property of the hamiltonian is its *functional form*, not its numerical value. Lots of functions have the same numerical value, but they don’t have the right functional form. They are not hamiltonians.

Summarizing: the real hamiltonian (4.5) encodes *all* the dynamics of the system, and gives *general* equations of motion that are always valid. To find a particular trajectory, one must supply auxiliary conditions (such as initial position and velocity), and solve for the *particular* motion. Our bogus “hamiltonian” was arrived at by solving *one* specific set of initial conditions, and then plugging the resulting *particular* solution of motion back into the real hamiltonian. The result is just gibberish; that is not a valid procedure for finding “another” hamiltonian.

An example closer to our topic is Figure 4.4d, which illustrates a classical electromagnetic system, similar to [Jaffe 2005 p2]. Here, we define our “system” to be a small mobile test charge distribution, and the electric field over all space. The fixed source charge is just an external potential for our system. The dynamic variables are the mobile charge location  $\mathbf{r}(t)$ , and the electric field throughout time and space  $\mathbf{E}(t, \mathbf{r})$ . The hamiltonian is:

$$H(\mathbf{p}, \mathbf{r}, V, \mathbf{A}) = \frac{(\mathbf{p} - e\mathbf{A}/c)^2}{2m} + qV(\mathbf{r}) + \int_{\infty} dVol \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi},$$

where  $\mathbf{E}$  and  $\mathbf{B}$  derive from  $V$  and  $\mathbf{A}$  in the usual way ( $\mathbf{E} = -\nabla V$ ,  $\mathbf{B} = \text{curl } \mathbf{A}$ ).

As the mobile charge moves, both  $\mathbf{r}(t)$  and  $\mathbf{E}(t, \mathbf{r})$  change, and given some auxiliary conditions, we can solve for them. However, plugging such a result back into the hamiltonian (replacing  $\mathbf{E}$  and  $\mathbf{B}$ ) gives nonsense. That is essentially the procedure used to “derive” the Casimir force from electromagnetic ZPE [Nikolic 2016]. This procedure gives the right answer in only one very special case (perfect conductors) of all the realistic cases, and that is just bad luck. (The Bohr model of the atom gives the right coarse energy levels, but the model is completely wrong.) The error in this part of the ZPE development of the Casimir force is that it circularly uses the *result* of the EM field cancellation to be the *cause* of the EM field cancellation.

**Extension to realistic situations:** [Gründler 2013] demonstrates convincingly that the vacuum energy approach to the Casimir force cannot be generalized to real plates having finite conductivity. Attempting to do so necessarily invalidates the original argument: essentially, realistic plates have currents, and the EM fields have non-sharp boundary conditions. This requires considering interactions between the fields and charges, violating the assumption that the force is a property of the fields alone and *independent* of charges. Without going into detail, we concur. We note that similarly, vacuum fluctuations cannot explain the force between dielectric plates, or dielectrics at finite temperatures, or dielectrics at short distances (where  $F \sim a^{-3}$ ) [Lifshitz 1956], [Schwinger+ 1978].

The ZPE Casimir development is either all or nothing. Which in this case, means it is nothing.

### Comments on the Casimir Effect

Here again, we see that the original work of [Ca&Po 1948] had no thought of ZPE, and only later did Casimir “find” a way to frame an argument in ZPE terms.

In [Lifshitz 1956] and [Schwinger+ 1978] are more general derivations of the Casimir force for dielectrics, using only QED, with no reference to vacuum fields. The force for conducting plates is a special case of that for dielectrics.

Multiple sources note that “Underlying the explanations of Casimir effects in terms of zero-point energy is a particular and arbitrary choice for the ordering of field operators,” [Milonni 1999 p203]; and from [M&S 2010 p42b], “We redefine the Lagrangian density  $\mathcal{L}$  and all observables ... as normal products. We are free to do this, as it merely corresponds to a particular order of factors before quantization. With observables defined as normal products, their vacuum expectation values vanish. In particular ...” the vacuum energy never appears. Once again, we must ask, “How do we choose the order of operators in our hamiltonian?” Since the ZPE development of the Casimir force is invalid with *any* hamiltonian, it provides no incentive to use the hamiltonian leading to ZPE.

Many laudable references have indeterminate positions on vacuum fields. For example, [Ma&Wo p508b] says, “Certain physical effects are sometimes attributed to vacuum fluctuations,” which suggest vacuum fields are optional. But then later, [Ma&Wo p512b] says, “Once again we see that the vacuum field plays a fundamental role and is required for internal consistency. ...and it cannot be ignored.” We disagree.

Similarly, [Milonni 1999] summarizes his Casimir analysis with, “The conclusion from this straightforward calculation is that we can derive the Casimir force using only fields due to *sources*. We do not have to invoke zero-point energy of the field or vacuum fluctuations” [emphasis his]. Surprisingly, he then concludes that the Casimir force provides “proof of a non-trivial vacuum field.” He goes on, “but that isn’t the only way to think about it.” Well then, that’s not proof.

#### 4.1.9 Conclusions on EM Vacuum Fields

Experimentally, there are no vacuum fields: there is no spontaneous absorption, the spontaneous emission rate excludes vacuum “stimulation,” the Lamb shift is stable (not variable), vacuum fluctuations cannot drive an infinite energy heat engine. All of these empirical results contradict vacuum fields. Theoretically, constants in the hamiltonian cannot be observed, uncertainty and superpositions are not fluctuations, Feynman diagrams do *not* imply fluctuations, vacuum fields cannot derive the Casimir force (for either ideal or realistic plates). Not only are vacuum fields unnecessary to QED, they are *precluded* by QED from any observable effects.



For vacuum fields to be real, there must be some larger theory that fully reproduces all results of QED, and *additionally* includes observable consequences of vacuum fields.

In the end, we cannot use the notion of vacuum fields to reliably estimate any phenomenon, because such estimates are often wrong, and their validity cannot be determined without a proper QED treatment. Instead, we must always use QED to reliably compute observable outcomes. From [Sc&Zu 1997]: “To be sure, the Welton treatment is no replacement for QED” [p607t]; “We stress, however, that the present heuristic approach [of vacuum fluctuations] is no substitute for a full QED analysis. In fact, stochastic electrodynamics provides answers contradicting QED for certain problems” [p609b].

The common error in reasoning about the vacuum is (1) starting with *classical* formulas, (2) demanding that *quantum* operators follow the classical formulas, and (3) insisting that the resulting unobservable constants have physical meaning. Instead, one should search for the QED hamiltonian that (1) satisfies experiment, (2) reduces to classical formulas in the appropriate limits, and (3) is as simple as possible. Then vacuum fields never arise. We have focused here on the EM field.

In our brief survey of the literature, some patterns emerge. First, nearly everyone with a detailed analysis concludes that vacuum fields do not exist. (One possible exception is [Milonni 1999], see earlier quote). Another pattern, this one historical, is that of making a discovery with well-established physics, and only *later* looking for and “finding” a way to describe it in vague, vacuum-field terms. Vacuum fields seem to always be an afterthought, rather than a reliable physical basis.

Some authors claim [e.g., Milonni 1999] that vacuum fields are necessary to preserve the canonical commutation relations between field operators, and are therefore necessary for consistency of the theory. Such claims seem implausible: the canonical commutation relations are usually the *starting point* for QFT, and are therefore inescapably built into the theory. (Milonni’s argument for the necessity of vacuum fields seems to compute a *particular* solution of the field equations, and then plug those results back into the commutators. But that should never have any effect on commutators that are a constant, such as  $i\hbar$ , which are independent of the quantum state of the field.) [Gründler 2013 p13] also seems to think the canonical commutation relations are a problem for QFT with no vacuum fields, but we have shown above that the normal-ordered hamiltonian has no effect on canonical commutations.

Finally, on the cosmological constant: the non-existence of vacuum EM fields, and the extreme quantitative disagreement between alleged ZPE and the astronomical measurement, strongly suggest that the cosmological constant is unrelated to ZPE. Therefore, attempts to relate the two are unlikely to succeed. However, there are claims that Higgs-sector physics is different than QED, and that it could have observable ZPE. We take no position on this.

#### 4.1.10 References

- [Bethe 1947] H. A. Bethe, “The Electromagnetic Shift of Energy Levels,” *Physical Review*, Volume 72 Number 4, August 15, 1947.
- [Be&Sa 1957] Bethe, Hans A. and Edwin E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms*, Springer-Verlag, 1957.
- [Ca&Po 1948] H. B. G. Casimir and D. Polder, “The Influence of Retardation on the London-van der Waals Forces,” *Physical Review*, Volume 73 Number 4, February 15, 1948.
- [Casimir 1948] H. B. G. Casimir, “On the attraction between two perfectly conducting plates,” *Proc.Kon. Nederl. Akad.Wetensch.* 51, 793 (1948), communicated at the meeting of the Royal Netherlands Academy of Arts and Sciences, May 29, 1948.
- [Casimir 1999] H. B. G. Casimir, “On the History of the So-Called Casimir Effect,” *Comments on Modern Physics*, Vol 1, Part D, 1999.
- [Gründler 2013] Gerold Gründler, “The Casimir-Effect: No Manifestation of Zero-Point Energy,” [arXiv:1303.3790v5](https://arxiv.org/abs/1303.3790v5) [physics.gen-ph], 2013.
- [Jaffe 2005] R. L. Jaffe, “Casimir effect and the quantum vacuum,” *Phys. Rev. D* 72, 021301(R) 12 July 2005.



- [Lifshitz 1956] E. M. Lifshitz, “The Theory of Molecular Attractive Forces between Solids,” *Journal of Experimental and Theoretical Physics*, January 1956.
- [Ma&Wo 1995] Leonard Mandel and Emil Wolf, *Optical Coherence and Quantum Optics*, Cambridge University Press, 1995. ISBN: 0 521 41711 2.
- [Milonni 1999] Peter W. Milonni, “Do Casimir Forces Imply a Non-trivial Vacuum,” *Comments on Modern Physics*, Vol 1, Part D, 1999.
- [Nikolic 2016] Hrvoje Nikolic, “Proof that Casimir force does not originate from vacuum energy,” *Physics Letters B*, Elsevier, 2016-08-18 p197.
- [Schwinger+ 1978] Julian Schwinger, Lester L. Deraad, Jr., and Kimball A. Milton, “Casimir Effect in Dielectrics,” *Annals of Physics* 115, pp1-23, 1978.
- [Welton 1948] Theodore A. Welton, “Some Observable Effects of the Quantum-Mechanical Fluctuations of the Electromagnetic Field,” *Physical Review* 74, 1157, 1 November 1948.

## 5 Appendices

### 5.1 References

#### 5.1.1 Books

- [Be&Sa 1957] Bethe, Hans A. and Edwin E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms*, Springer-Verlag, 1957.
- [M&S 2010] F. Mandl and G. Shaw, *Quantum Field Theory*, 2<sup>nd</sup> ed, John Wiley & Sons, Ltd, 2010. ISBN: 978-0-471-49683-0.
- [Ma&Wo 1995] Leonard Mandel and Emil Wolf, *Optical Coherence and Quantum Optics*, Cambridge University Press, 1995. ISBN: 0 521 41711 2.
- [P&S 1995] Peskin, Michael E. and Dan V. Schroeder, *An Introduction to Quantum Field Theory*, Westview Press October 2, 1995, ISBN-13: 978-0201503975.
- [Sc&Zu 1997] Scully, Marlan O. and M. Suhail Zubairy, *Quantum Optics*, Cambridge University Press, 1997, ISBN 0 521 43595 1.

#### 5.1.2 Articles

- [Lif 1956] E. M. Lifshitz, "The Theory of Molecular Attractive Forces between Solids," *Journal of Experimental and Theoretical Physics*, January 1956.