Funky Relativity Concepts

The Anti-Textbook*

A Work In Progress. See elmichelsen.physics.ucsd.edu/ for the latest versions of the Funky Series.
Please send me comments.

Eric L. Michelsen

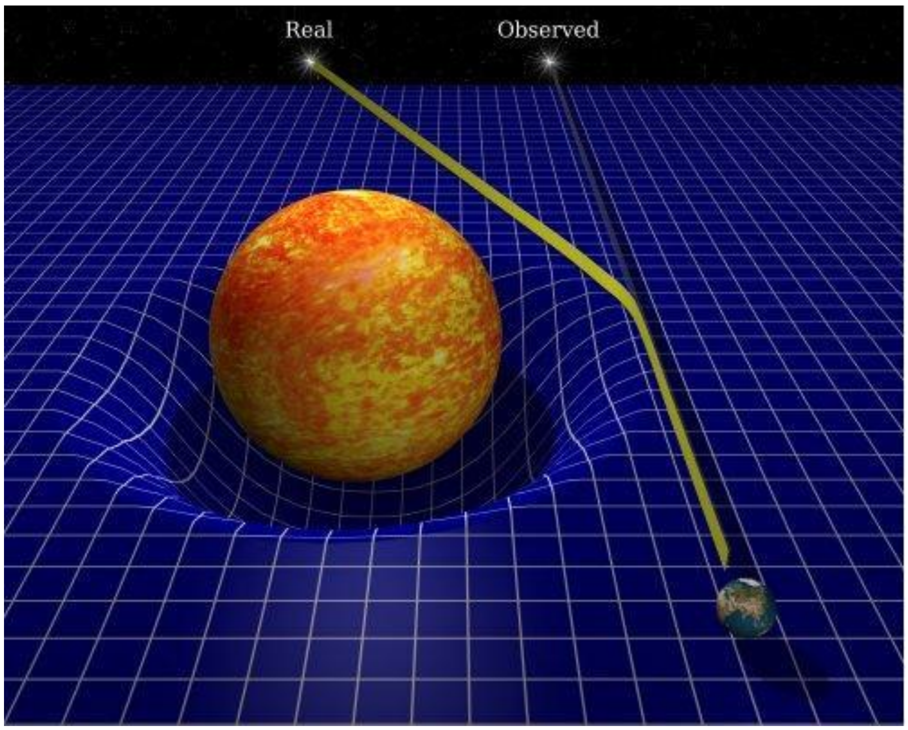

Image Source: http://www.space.com/. Need better graphic??

"A person starts to live when he can live outside himself."

"Weakness of attitude becomes weakness of character."

"We can't solve problems by using the same kind of thinking we used when we created them."

"Once we accept our limits, we go beyond them."

"If we knew what we were doing, it would not be called 'research', would it?"

--Albert Einstein

* Physical, conceptual, geometric, and pictorial physics that didn't fit in your textbook.

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Physical constants: 2006 values from NIST. For more, see http://physics.nist.gov/cuu/Constants/ .

Gravitational constant	$G = 6.674\ 28(67) \times 10^{-11}\ m^3\ kg^{-1}\ s^{-2}$
Relative standard uncertainty	1.0×10^{-4}
Speed of light in vacuum	$c = 299\ 792\ 458\ m\ s^{-1}$ (exact)
Boltzmann constant	$k = 1.380\ 6504(24) \times 10^{-23}\ J\ K^{-1}$
Stefan-Boltzmann constant	$\sigma = 5.670\ 400(40) \times 10^{-8}\ W\ m^{-2}\ K^{-4}$
Relative standard uncertainty	$\pm 7.0 \times 10^{-6}$
Avogadro constant	$N_A, L = 6.022\ 141\ 79(30) \times 10^{23}\ mol^{-1}$
Relative standard uncertainty	$\pm 5.0 \times 10^{-8}$
Molar gas constant	$R = 8.314\ 472(15)\ J\ mol^{-1}\ K^{-1}$
calorie	4.184 J (exact)
Electron mass	$m_e = 9.109\ 382\ 15(45) \times 10^{-31}\ kg$
Proton mass	$m_p = 1.672\ 621\ 637(83) \times 10^{-27}\ kg$
Proton/electron mass ratio	$m_p/m_e = 1836.152\ 672\ 47(80)$
Elementary charge	$e = 1.602\ 176\ 487(40) \times 10^{-19}\ C$
Electron g-factor	$g_e = -2.002\ 319\ 304\ 3622(15)$
Proton g-factor	$g_p = 5.585\ 694\ 713(46)$
Neutron g-factor	$g_N = -3.826\ 085\ 45(90)$
Muon mass	$m_\mu = 1.883\ 531\ 30(11) \times 10^{-28}\ kg$
Inverse fine structure constant	$\alpha^{-1} = 137.035\ 999\ 679(94)$
Planck constant	$h = 6.626\ 068\ 96(33) \times 10^{-34}\ J\ s$
Planck constant over 2π	$\hbar = 1.054\ 571\ 628(53) \times 10^{-34}\ J\ s$
Bohr radius	$a_0 = 0.529\ 177\ 208\ 59(36) \times 10^{-10}\ m$
Bohr magneton	$\mu_B = 927.400\ 915(23) \times 10^{-26}\ J\ T^{-1}$

Other values:

1 inch ≡ 0.0254 m (exact)

1 drop ≡ .05 ml (metric system, exact. Other definitions exist.)

1 eV/particle = 96.472 kJ/mole

1 esu ≡ 1 statcoulomb = $3.335\ 641 \times 10^{-10}$ C

kiloton ≡ 4.184×10^{12} J = 1 Teracalorie

bar ≡ 100,000 N/m^2

atm ≡ 101,325 N/m^2 = 1.013 25 bar

torr ≡ 1/760 atm ≈ 133.322 N/m^2

Contents

Figures

[I'm testing the utility and appearance of a Table of Figures.]

1 Introduction

Why Relativity?

Relativity is relevant in a growing number of earthly endeavors. Special Relativity is important for high-field magnetized plasmas (say, for fusion energy), GPS, electron beams, and more. Astrophysical studies often rely on effects of General Relativity for black holes, neutron stars, gravity waves, cosmology, etc.

How to Use This Document

If you're a serious student of General Relativity (GR) who needs a clear, concise summary of unfamiliar relativity concepts, then this book might be for you. Many concepts are consistently unclear in most of the popular references. The Funky series attempts to clarify those neglected concepts, and others that seem likely to be challenging and unexpected (funky?). But *this work is not a text book*; there are plenty of those, and they cover most of the topics quite well. This work is meant to be used *with* a standard text, to help emphasize those things that are most confusing for students new to relativity. When standard presentations don't make sense, come here.

If you don't understand something, read it again *once*, then keep reading.
Don't get stuck on one thing. Often, the following discussion will clarify things.

You should read all of this introduction to familiarize yourself with the notation and contents. After that, this work is meant to be read in the order that most suits you. Each section stands largely alone, though the sections are ordered in a logical sequence. You may read it from beginning to end, or skip around to whatever topic is most interesting.

The index is not yet developed, so go to the web page on the front cover, and text-search in this document.

The Funky Series

The purpose of the "Funky" series of documents is to help develop an accurate physical, conceptual, geometric, and pictorial understanding of important physics topics. We focus on areas that don't seem to be covered well in any text we've seen. The Funky documents are intended for serious students of physics. They are not "popularizations" or oversimplifications, though they try to start simply, and build to more advanced topics. Physics includes math, and we're not shy about it, but we also don't hide behind it.

Without a conceptual understanding, math is gibberish.

This work is one of several aimed at graduate and advanced-undergraduate physics students. I have found many topics are consistently neglected in most common texts. This work attempts to fill those gaps. It is *not* a text in itself. You must use some other text for many standard presentations.

What's Wrong With Existing Relativity Expositions?

They're not precise enough with their definitions. Usually, when there appears to be an obvious contradiction, it is a confusion of definitions. Many widely used references have surprisingly unclear definitions, and one purpose of this work is to help resolve them. Also, they're not visual or graphical enough. They rely way too much on algebra or advanced math, and not enough on insight.

My Story

The Funky series of notes is the result of my going to graduate school in physics after 20 years out of school. There are many things I wish I had understood better while taking my graduate physics classes (first at San Diego State University, then in the PhD program at University of California, San Diego). Although I had been an engineer all that time, most of my work involved software and design architectures that are far removed from fundamental science and mathematics. It's clear that many professors have forgotten what it's like to be learning these concepts for the first time. I've tried to write most of these notes as I learn them, so that I remember all the stumbling blocks, and can clarify them.

Thank You

I owe a big thank you to many professors at both SDSU and UCSD, for their generosity even when I wasn't a real student: Dr. Peter Salamon, Dr. Arlette Baljon , Dr. Andrew Cooksy, Dr. George Fuller, Dr. Tom O'Neil, Dr. Terry Hwa, and others.

Scope

What This Text Covers

This text covers some of the unusual or challenging concepts in relativity, from Special Relativity through a first graduate course in General Relativity (GR). It is also very suitable for undergraduate GR, as well. We expect that you are taking or have taken such a course, and have a good text book. This text supplements those other sources.

What This Text Doesn't Cover

This text is not a relativity course in itself, nor a review of such a course. We do *not* cover all basic relativity concepts; only those that are neglected, unusual, or especially challenging (funky?).

What You Already Know

This text assumes you understand basic integral and differential calculus, and partial differential equations. Some sections require a working knowledge of a related physics topic, such as electromagnetics. We assume you have a relativity text for the bulk of your studies, and are using *Funky Relativity Concepts* to supplement it.

Notation

TBS stands for "To Be Supplied," i.e., I'm working on it. Let me know if you want it now.

?? For this work in progress, double question marks indicates areas that I hope to further expand in the final work. Reviewers: please comment especially on these areas, and others that may need more expansion.

Keywords are listed in **bold** near their definitions. All keywords also appear in the glossary.

Formulas: Evaluation between limits: we use the notation $[\text{function}]_a^b$ to denote the evaluation of the function between a and b, i.e.,

$[f(x)]_a^b = f(b) - f(a)$. For example, $\int_0^1 3x^2\, dx = [x^3]_0^1 = 1^3 - 0^3 = 1$

We write the probability of an event as "Pr(event)."

Open and closed intervals: An open interval between c and d is written (c, d). It means the range of numbers from c to d *exclusive* of c and d. A closed interval between c and d is written $[c, d]$. It means the range of numbers from c to d *including* c and d. A half-open interval $[c, d)$ has the expected meaning of c to d including c but not d, and $(c, d]$ means c to d excluding c but including d.

Vector variables: In some cases, to emphasize that a variable is a vector, it is written in bold; e.g., V(**r**) is a scalar function of the vector, **r**. **E**(**r**) is a vector function of the vector, **r**.

In my word processor, I can't easily make fractions for derivatives, so I sometimes use the standard notation d/dx and $\partial/\partial x$.

I've never understood the bother about distinguishing between d/dx and $\partial/\partial x$. When the function arguments are independent, both forms of derivative are obviously the same thing; I don't know why there's even two ways to write it. Nonetheless, only as a matter of convention, I use d/dx when a function is clearly a total derivative, and $\partial/\partial x$ when it is clearly a partial derivative. However, in some cases, it's not clear what arguments a function has, and it's not important. In that case, I tend to use $\partial/\partial x$ for generality, but don't worry about it.

And for the record, derivatives *are* fractions, despite what you might have been told in calculus. They are a special case of fraction: the limiting case of differentially small changes. But they are still fractions, with all the rights and privileges thereof. Everyone treats them like fractions, multiplies and divides them like fractions, etc., because they *are* fractions. This is especially relevant to differential geometry.

> Interesting points that you may skip are "asides," shown in smaller font and narrowed margins. Notes to myself may also be included as asides.

Ideas for a General Relativity Course

I would teach things differently than most I've seen.

- I would have to start with Special Relativity, because most graduate students have less of an understanding than they realize. Some sample spacetime diagrams for things like the pole in the barn, and the twin effect.
- I would emphasize the distinction between a reference frame and a coordinate system. While all reference frames can be used to define a coordinate system, not all coordinate systems are physically possible reference frames. This is actually relevant later in the Kerr black hole, where the natural coordinates, at some places, move at the speed of light, and there the coordinate velocity of light is zero. I watched a whole class of graduate students be confused by this, and nearly all finished the course without ever understanding it. This distinction also helps explain why the superluminal speeds measured by a rotating observer at large distances are not a problem.
- I'd show how special relativity already implies the curvature of space, even before GR was discovered (observers on and off a merry-go-round). This also shows explicitly that spacetime curvature is an invariant *only for inertial observers*.
- I would emphasize how curvature of spacetime looks like our intuitive notion of gravity (e.g., MTW's "Curvature of What?" box). I would emphasize that the current trend of using the term "curvature of time" makes no sense, since curvature can only be defined on (sub)manifolds of 2 or more dimensions (despite some references that use this term). Familiar gravity comes from the curvature of time-space submanifolds. I'd show why light doesn't fall like matter does, but actually twice as fast.
- What causes gravity? I would discuss the sources of gravity early, even before the field equations. Schutz's "Gravity from the Ground Up" has the best I've seen of this, but it is still (in my view) lacking. Most courses never actually cover the real meaning of the stress-energy tensor (the conserving currents for energy and the 3 components of momentum). I might write the field equations, but not focus on them yet.
- I'd introduce differential geometry. Discuss the difference between topology (global description), and geometry (local measurements). I'd include the discussion of the topology and geometry of the donut.
- I would then discuss the meaning of the metric, amplifying on how curvature looks like a force. In particular, I'd include gravitomagnetism, and show how the first row (or column) of the metric is the vector potential for the gravitomagnetic field. Most courses ignore this completely, yet it is not only important, but was recently a hot area of discussion (APOLLO's research was involved in this controversy). We need more light on this. Explain why the term "frame dragging" is a terrible misnomer, and why we should use "gravitomagnetism" instead.
- I'd include some real-world calculations of, say, gravitational time dilation of GPS satellites (which requires a non-obvious use of both SR and GR), and perhaps the geodetic precession of polar orbiting satellites (which is also measured).
- Then I'd discuss the field equations in more detail. Schwarzschild solution. The real meaning of the coordinates inside the horizon. You have to rename them: '*t*' goes to '*a*', which is spatial, and '*r*' goes to '*w*', which is time. Students simply cannot get onboard with the notion that '*t*' is spatial, and not time. I'd talk about whether it is even science to discuss the interiors of black holes, since no experiment, even in principle, can reveal what's inside. It is a non-testable (non-falsifiable) theory, which is a term often used to describe religion.
- I'd discuss how curvature is only invariant for inertial observers; accelerating observers measure curvature differently (as exemplified by the acceleration-dependent horizon behind accelerating observers, or by any revolving observer).
- I might talk about some of the oddities of the Kerr black hole: their coordinates that move at the speed of light, the two horizons, the closed time-like curves (that no one understands).

Possible Future Topics

1. Suppose we're in a closed universe. Alice flies by Bob when they are the same age. She continues unaccelerated around the entire universe until she returns to Bob. Who is older?

2 Special Relativity

Introduction to Special Relativity

Much of modern physics relies on the physics of Special Relativity (SR). Without SR, modern physics would not exist. SR describes the dynamics (i.e., forces and accelerations) of bodies at high speeds, $v \sim c$. Like Newtonian mechanics, SR applies to **inertial** observers: observers who feel no acceleration. If an inertial observer releases an object from her hand, and no forces act on it, she sees the object remain motionless. Note that

> In SR, the *objects* we observe accelerate when pushed by forces; that's the definition of "dynamics." It's the *observers* that must be inertial to use the laws of SR.

Special Relativity (SR) is introduced in *Funky Electromagnetic Concepts*. New students should start there. Here, we set the stage with some concepts, and then go into more detail about Special Relativity. We assume you are familiar with the basics of establishing a valid inertial reference frame, synchronizing clocks, and making measurements of distance and time. Note that observers exist in "reference frames" (described below).

Terminology: When we say (loosely) an observer "sees" something, we mean the observer "measures" something. Reality is what we measure. We *do not* mean that the observer literally "sees" something that is an optical illusion, due to the propagation delay of light. We are *not* concerned here with such illusions; we are concerned with self-consistent physical measurements.

> In my opinion, the *only* axiom of Special Relativity is that
> the laws of physics are the same for all inertial (unaccelerated) observers.

The constancy of the speed of light is a *consequence* of this statement [Sch p181; Car p??]: electromagnetics sets the speed of light; electromagnetics is the same for all unaccelerated observers; therefore, the speed of light is the same for all unaccelerated observers.

> We will see that in General Relativity, where observers can be accelerated,
> the speed of light is *not constant*, and the whole concept of "speed" is more subtle.

Special Relativity Basic Concepts

We introduce here several important examples of Special Relativity physics that emphasize fundamental concepts, and require little math. These concepts underlie the subsequent, more quantitative analyses.

Relativity Implies New Dynamics, Even for a Single Observer

Part of the study of relativity concerns how *different* observers will measure the *same* events: given measurements in one frame, we can compute comparable measurements in another. But an even more important consequence of relativity is:

> Even when considering physics from only *one* frame of reference,
> relativity requires new dynamics (post-Newtonian).

For example, the definitions of energy and momentum change. Relativity is important for high-speed physical computations.

Why Does Relativity Talk About Astronauts So Much?

TBS: the earth creates an illusion of "stationary" and "absolute." We're so used to the solid, unstoppable power of the earth, that it's hard to free ourselves from the prejudice it creates. That's why the train thought-experiment of Einstein is hard to follow. He supposed an observer on a moving train, and an observer on earth. They are equivalent, but it is hard to avoid thinking the earth-observer is "right", and the train observer is misled. Instead, becoming an astronaut in space, far from any large body, frees us to understand the true relative nature of observations.

Mass Is Invariant

The modern view is that mass is a Lorentz invariant: all observers measure the same mass.

It is the momentum which includes the γ factor (rather than the mass):

$$\mathbf{p} = \frac{1}{\sqrt{1-(v/c)^2}} m\mathbf{v} \qquad \text{where} \quad v \equiv |\mathbf{v}|$$

You may have heard that mass increases with speed. That is an old view, no longer used.

The old view that mass changes with speed doesn't work very well. Motion can be decomposed into components in 3 directions, which leads to obtuse concepts of "transverse mass" and "longitudinal mass." It's not that older physicists got the physics wrong; it's just that the old theory is unnecessarily complicated. The new theory is simpler.

Note that the term "rest-mass" is an anachronism from the old view. Today, "mass" and "rest-mass" mean the same (invariant) thing.

Coordinates and Reference Frames

Not All Coordinate Systems Are Reference Frames

Coordinates and reference frames underlie nearly all of relativity analysis. The two are often confused, because they often can be thought of together. But not always. The distinction is important.

We choose coordinates for mathematical convenience,
and reference frames for pedagogical convenience.

A **coordinate system** is just a way to label spacetime points. For example, we could choose Solar System Barycenter coordinates (SSB), in which the center of mass of the solar system is fixed in space over time, and the coordinates are stationary with respect to the distant stars. In this system, the earth's spatial coordinates vary with time, because the earth moves with respect to the solar system center of mass. If I'm studying the orbit of the planets, and how they are affected by each other, this is by far the simplest choice of coordinates. Note that the sun also moves a little in the SSB frame, since it must offset the positions of the planets. For precision measurements, this small movement matters.

Alternatively, we could choose earth-centered coordinates, in which the center of mass of the earth is fixed over time. Even with this choice, we have freedom to choose more: how do the spatial coordinates rotate with time? If I am planning to drive across the country, I will probably want a coordinate system in which not only the center of mass of the earth is fixed, but the earth's orientation is fixed. I don't care about the earth's rotation when I'm driving cross-country. The standard longitude/latitude coordinates is such a system.

On the other hand, if I'm designing the GPS system, with orbiting satellites, I probably want a coordinate system centered on the earth, with orientation fixed with respect to the distant stars. In this system, the satellite orbits are simplest, and they are not (much) affected by the earth's rotation. The earth rotates underneath the satellites, and earthlings see them constantly rising and setting throughout the day. The celestial orientation separates the satellite orbits from the earth's rotation, and each can be calculated without regard for the other.

Separately, a **reference frame** starts with a state of motion in which a massive observer could exist. The reference frame then defines coordinates stationary with respect to such an observer. As with most physics concepts, for pedagogy, we use an idealization of an observer as some single, indivisible thing.

[Aside: Of course, a real observer is generally a composite of zillions of particles, whose relative motion is almost always too small to matter. Strictly speaking, though, if my machine detects a photon, it is the relative motion of the photon and the smallest part of the machine with which it interacts that matters. If the machine converts the photon to an electric pulse, then the rest of the machine which measures the pulse is not relevant to the original photon interaction.]

Some coordinate systems can be reference frames: if a massive observer can remain at fixed coordinates over time, then the coordinate system defines a reference frame. However, it is possible to define a coordinate system which moves faster than light relative to some body. Coordinates are just numeric labels of spacetime; we can define them any way we want. By themselves, they don't describe any physical motion; they're just labels. In some cases, such fast-moving coordinates are more sensible than it might seem. For a rotating black hole, we use coordinates that solve Einstein's equation most simply. In that system, at the event horizon, light travels around the black hole with constant coordinates over time. In other words, the coordinate system moves at the speed of light, following along with the light itself. If you could move along with the coordinate system, the light would look stationary. But you can't: massive bodies *always* travel slower than light. This coordinate system does *not* define a reference frame, because no observer can be "stationary" in it.

All reference frames can be used to define coordinate systems,
but not all coordinate systems can be used as reference frames.

We will return to reference frames in General Relativity.

Inertial Reference Frames Are Idealized

A special case of reference frame is one which is inertial: bodies at rest stay at rest, and bodies in motion move in a straight line at constant speed. However:

In the universe, there are no finite, exactly inertial reference frames.

That's because there are massive bodies all around us, and everything in any reference frame is pulled by their gravity. We can approximate inertial reference frames in various ways. One way is to fall freely. Then the bulk of gravity is invisible to us, but we can still see tiny tidal forces (differences in the strength of gravity with position).

As another nearly-inertial frame, we can approximate weak gravity as a force (vs. in GR, where gravity is spacetime curvature instead of a force). Therefore, we can approximate observers in weak gravity as inertial, but with all bodies subject to the external force of gravity.

Or, we can restrict ourselves to a two-dimensional surface, such as a table top, where motion in the direction of gravity is blocked. Then we have a 2D inertial reference frame.

All 3 of the reference frames at the beginning of this section, SSB, longitude/latitude, and earth-centered/star-oriented, are approximately inertial frames. The longitude/latitude frame is inertial in 2D parallel to the earth's surface, or in 3D if we consider every body is acted on by an external force of earth's gravity. The other two are approximately inertial because gravity is weak, and we can approximate it as a force.

Therefore, in principle and in practice, we can construct nearly-inertial reference frames, with high accuracy. As in all physics, then, we use the idealized concept of a truly inertial reference frame as a tool for understanding, even though it can never be exactly realized.

How to Construct a Valid Inertial Reference Frame

See *Funky Electromagnetic Concepts* for details on how to construct a valid reference frame. The key points are that the inertial frame has clocks scattered all over it, wherever they are needed. All the clocks are synchronized. The "time" of any event is measured by a clock right at the event. Therefore, the speed of light does *not* play a role in any measurement. In particular, we are careful to distinguish what an observer *measures* from what a viewer might "see" with her eyes.

The act of seeing is subject to light propagation delays; measurements are not.

How do I test if my frame is inertial? TBS: Release 3 particles at rest at 3 non-collinear points. If they all remain stationary, my frame is inertial. If any particle accelerates, my frame is *not* inertial.

Simple Examples of Special Relativity

Do Things Really Shrink, and Clocks Slow Down, When They Move?

This is a little tricky, so a one-word answer is likely to mislead. Things don't change when they move, or when the observer moves. For example, suppose Alice and I are astronauts floating at rest around our rocket. I measure its length. Then I fire a quick blast of my backpack jets, so I am now moving relative to the rocket at constant speed (I am *not* now accelerating). Nothing has happened to the rocket. It hasn't accelerated, nor "moved" from Alice's point of view. She measures it the same length as before. However, I now measure the rocket shorter. In my reference frame, *the rocket really is shorter*. My measurements are valid and self-consistent; there's nothing wrong with them. But they are different than Alice's measurements. *The rocket has not shrunk; it is simply shorter in my reference frame*. My meters are different than Alice's.

Length is relative to the observer. There is no absolute distance.

Similarly, I now look at Alice's wristwatch, and compare it to my own. I measure that her watch runs slower than mine. In my reference frame, it *really does* run slower. Alice measures no change in her wristwatch. *Alice's watch has not slowed down; it simply runs slower in my reference frame*. My seconds are different than Alice's.

Time is relative to the observer. There is no absolute time.

How Far is Far?

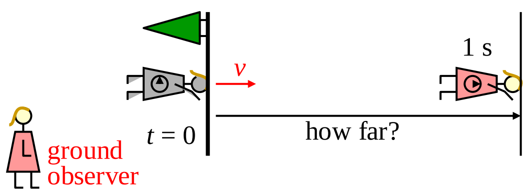


Figure 2.1: How far do I go in 1 s at 0.5c?

Suppose I travel at 0.50c (Figure 2.1). Starting at a flag stuck in the ground, I travel for 1 s by my watch, then stop (such that my deceleration time is negligible). How far am I from the flag? The simplistic answer is wrong:

$$0.50(3.0\times10^8 \text{ m/s})(1 \text{ s}) = 1.5\times10^8 \text{ m} \qquad (\textit{wrong answer}) .$$

In fact, just before decelerating, I *do* measure that I am 1.5×10^8 m from the flag. A ground observer measures ground distance longer than this, but I measure his length contracted by the factor:

$$\gamma = \frac{1}{\sqrt{1-.5^2}} = \frac{2}{\sqrt{3}} = 1.15$$

When I stop, I enter the ground frame. The distance is no longer length contracted, so my final distance to the flag is:

$$d = \gamma\left(1.5\times10^8 \ m\right) = 1.73\times10^8 \text{ m} .$$

Note that a ground observer stationary with the flag calculates the same result without length contraction, as follows: He measures my clock runs slowly by a factor γ. Therefore, 1 s on my watch is γ seconds on his. The distance is just time $\times$ speed:

$$d = (\gamma \text{ s})\left(1.5\times10^{8} \text{ m/s}\right) = 1.73\times10^{8} \text{ m}.$$

Example: Two identical charges are at rest, spaced 1 m apart. The charges are accelerated identically, say by a uniform electric field. The two charges are now moving in unison, at speed *v*. Is their separation now less than, equal to, or more than 1m? How about from the frame of little observers riding on the particles?

Answer: In the lab frame, both charges have been displaced identically from rest. They are at all times separated by 1m.

Observers on the charges (Chargers) hear that lab people measure the separation as 1m. But lab rulers are short, according to Charger measurements. Therefore, the Chargers find the distance between charges is less than one meter. TBS.

E = mc²

In 1905, Einstein computed the mass of electromagnetic energy. However, then Einstein reasoned that all energy must weigh the same, because inside a black box, the energy can be converted to different forms. If different forms of energy weighed differently, the mass of the box could change without any outside interaction. A similar argument will come up again in an important way with angular momentum.

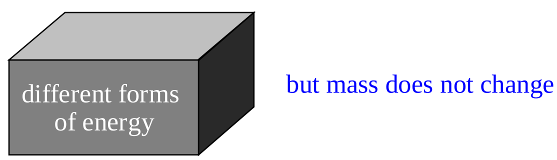


Figure 2.2: Mass doesn't change, even if the form of energy inside does.

What's Valid in Relativity?

This section assumes a basic understanding of special relativistic dynamics (SR). You should understand frames of reference (aka "observers"), and the following:

$$E = \gamma mc^2 \qquad E^2 = c^2p^2 + c^4m^2 \qquad KE = (\gamma - 1)mc^2 \qquad \mathbf{p} = \gamma m\mathbf{v}$$

	Non-relativistic	**Relativistic**
Equation of motion	$F \equiv dp/dt$	same (definition of force)
Equation of motion (1D)	$F = ma$	$F = m\gamma_v^3 a$
Reaction force (3rd "law" for non-magnetic forces)	implies conservation of momentum	same
Conservation of momentum	$p \equiv mv$	$p \equiv \gamma mv$
Energy	$KE = ½\, mv^2$	KE + rest-energy = γmc^2
Conservation of energy	$E = KE + PE$	$E = KE$ + rest-energy + $PE = \gamma mc^2 + PE$
Work/energy	$dE = đW = F\,dx$	same
Magnetic force	$F = q\mathbf{v} \times \mathbf{B}$	same
Lagrangian	$L(x, v, t) =$ $T(x, v) - V(x, t) + q\mathbf{v}\cdot\mathbf{A}(x, t)$	$L(x, v, t) = -mc^2/\gamma - V(x, t) + q\mathbf{v}\cdot\mathbf{A}(x, t)$

Hamiltonian	$H(x_i, p_i, t) \equiv \Sigma p_i v_i - L(x_i, v_i, t)$	same definition of hamiltonian, but different L implies different H

Similarities (green) and differences (red) between non-relativistic and relativistic physics.

We derive the relativistic acceleration due to a 3-force from the definition $F \equiv dp/dt$. In the direction of motion:

$$F \equiv \frac{dp}{dt} = \frac{d}{dt}(\gamma_v mv) = m\underbrace{\left(\gamma^3 \frac{v}{c^2}\frac{dv}{dt}v + \gamma\frac{dv}{dt}\right)}_{\text{product rule}} = m\left(\gamma^3\frac{v^2}{c^2} + \gamma\right)a \quad where \quad a \equiv \frac{dv}{dt}$$

But $$\gamma^3\frac{v^2}{c^2} + \gamma = \gamma^3\left(\frac{v^2}{c^2} + \gamma^{-2}\right) = \gamma^3, \qquad \text{using } \gamma^{-2} = 1 - \frac{v^2}{c^2}$$

$$\Rightarrow \qquad F = m\gamma^3 a$$

Note that, quite differently from non-relativistic motion:

The relativistic acceleration from a given force depends on the current speed.
The faster the current speed, the less the acceleration.

Perpendicular to the motion: Note that *force* perpendicular to the motion does *not* mean *acceleration* only perpendicular to the motion. For example, if moving only in the x-direction, and force is only in the y-direction, then x-*momentum* doesn't change. But since there is acceleration in the y-direction, γ increases, so v_x must *decrease* for fixed x-momentum!

Acceleration in the y-direction causes *slowing* in the x-direction.

Derivation TBS.

What about work? Let us consider the relativistic change in energy from a force applied over a distance:

$$E^2 = c^2p^2 + c^4m^2 \qquad \text{Differentiate:}$$

$$2E\ dE = 2c^2p\ dp = 2c^2pF\ dt \qquad \text{Use:} \quad E = \gamma mc^2$$

$$\gamma mc^2\ dE = c^2pF\ dt$$

$$dE = \frac{pF}{\gamma m}dt = \frac{\gamma mvF}{\gamma m}dt = Fv\ dt = F\ dx.$$

This is the same work-energy theorem as NR.

Of Rockets and Relativity

How would we build a relativistic rocket? What kind of fuel should it use? What kind of particles should it exhaust? How does energy conversion efficiency factor in? What about ion engines?

Physics gives us some direction.

The fastest human-made rocket to date (2009) was the New Horizons craft sent to Pluto, and beyond, in 2006. It traveled 20,000 times slower than light (highly non-relativistic). However, it is quite plausible that ion engines might someday accelerate extra-solar-system spacecraft to relativistic speeds. If so, then understanding the relativistic rocket becomes a practical necessity. Furthermore, it is an interesting and highly informative study of special relativistic dynamics, so it is worthy of examination. There is a lot of confusion about the relativistic rocket, and different references arrive at different results (some are wrong). We describe the derivation in the simple case of no gravity, following these steps:

- Understanding mass, energy, and the special treatment given kinetic energy (KE)
- The non-relativistic (NR) rocket
- The relativistic rocket, including imperfect efficiency
- Further observations, and explanation of common mistakes

Common mistakes include incorrectly computing the mass of the fuel's potential energy (or neglecting it completely), and confusing the reference frames of the observer (say, on the ground) and the rocket.

This section assumes a basic understanding of special relativistic dynamics (SR). You should understand frames of reference (aka "observers"), and the following:

$$E = \gamma mc^2 \qquad KE = (\gamma - 1)mc^2 \qquad \mathbf{p} = \gamma m\mathbf{v}$$

Notation: As usual, we define the SR dilation factor, and its differential, as functions of $v \equiv |\mathbf{v}|$:

$$\gamma(v) \equiv \gamma_v = \left(1 - v^2/c^2\right)^{-1/2}, \qquad d\gamma_v = -\frac{1}{2}\left(1 - v^2/c^2\right)^{-3/2} \cdot \left(-2v/c^2\right)dv = \gamma_v{}^3\left(v/c^2\right)dv$$

Though γ is a function of v, we write v as a subscript γ_v, for readability in upcoming equations.

Critical Mass

The so-called "rocket equation" is a differential equation relating the speed of a rocket and the remaining mass of the rocket and fuel. Before tackling the relativistic rocket equation, we must review some fundamental concepts of mass and energy, as viewed in the modern formulation of special relativity.

In the old days, scientists commonly referred to the mass of objects "increasing" when they were moving. This is a way of incorporating the mass of the KE of the moving object. The modern view is arguably simpler and more consistent. Instead of the old mass as a function of speed, $m(v) = \gamma_v m_0$, we use only the rest mass of the object ($\equiv m$), and change our definitions of momentum and energy:

$$E = \gamma_v mc^2 \ \ (rest + kinetic), \qquad \mathbf{p} = \gamma_v m\mathbf{v} \qquad where \quad m \equiv \text{rest mass of object} . \qquad (2.1)$$

Therefore:

Mass is *defined* as rest-mass. It is a Lorentz scalar: it is the same for all observers.

In the modern view, mass does not change with speed.

The new definitions are most clearly superior to the old in the case of two- or three-dimensional motion. If a particle is moving in the *x*-direction, and we then accelerate it in the *y*-direction, the old definition of mass required *two* masses: "longitudinal mass" and "transverse mass" [ref??]. This was awkward. The new definitions (2.1) work for *all* motion in all dimensions.

As with NR (non-relativistic) motion, we also have:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \qquad but \qquad \mathbf{F} \ne m\mathbf{a} \quad \text{(relativistically)} .$$

However, it is still a fact that all energy has mass, given by the most famous equation in physics:

$$m = E / c^2 \qquad \text{(all forms of energy)} .$$

All forms of energy must have the same mass, because if we have a closed box with energy in it, and we convert the energy from one form to another, the mass of the closed box cannot change.

The mass of kinetic energy is the same as the mass of any other kind of energy.

So now we have a conundrum: we found it is more convenient *not* to include the mass of kinetic energy in the mass of a moving body: we say the "mass" is defined as *rest* mass. However, we also just said that *all* forms of energy, including kinetic, must have mass. This makes kinetic energy *special*: physicists choose to treat it differently than other energy, and to use the modern energy/momentum equations (2.1).

The modern energy/momentum equations are written in terms of rest-mass, and already include the effect of the mass of a body's kinetic energy. When a body contains potential energy (such as a compressed spring inside it, chemical energy, nuclear, electrical, etc.), that PE has mass $m = PE/c^2$, and is part of the rest-mass of the body.

In this section, we will lay out two analyses: one for exhaust that is the end-product of burning the energy source (fuel), and one for engines where the energy source is completely separate from the exhaust mass, such as an ion engine.

The Classical (Non-relativistic) Rocket

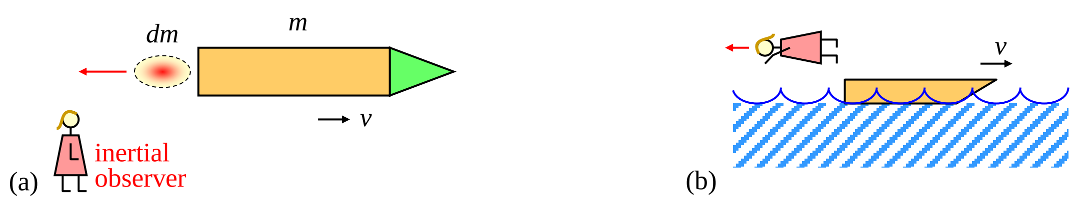


Figure 2.3: (a) The rocket question. (b) The similar question of "person diving off the boat".

Before tackling the relativistic rocket, we first summarize the classical rocket, to illustrate some principles. We seek the relation *v*(*m*) between the speed of the rocket and its remaining mass (after it has consumed some fuel). This is sometimes given in early physics as the "people diving off the boat" question: people jump off the back of a boat; how fast does the boat move forward?" (The relativistic rocket is the same principle with some complications.)

A rocket works by throwing mass out the back (exhaust) to increase the speed of the remaining rocket (rocket plus remaining fuel). Conservation of momentum allows us to compute the change in speed. Let $m \equiv$ mass of rocket (plus remaining fuel), which is the most relevant quantity for rocket motion. Consider a rocket moving through space, and a chunk of its exhaust, with mass $|dm|$ (Figure 2.3a). The rocket gets lighter as it exhausts fuel, so $dm < 0$. The momentum of the rocket plus exhaust is the same before and after ejecting the mass. The ejected mass has less momentum than when it was inside the rocket:

$$dp_{exhaust} = e\, dm \quad where \quad e \equiv \text{exhaust speed relative to rocket } (>0), dm < 0 .$$

Therefore, the remaining (rocket + fuel) momentum increases by this magnitude:

$$dp_{r+f} = m\, dv = e(-dm) \quad \Rightarrow \quad dv = -e\frac{dm}{m} . \qquad (2.2)$$

[A more tedious way of deriving this considers the total momentum of the rocket + fuel and the ejected mass (omit??):

$$p_{before} = p_{after} \quad \text{or} \quad \underbrace{\sum mv}_{\text{before}} = \underbrace{\sum mv}_{\text{after}} \quad \text{(non-relativistic)}.$$

$$\begin{aligned}
mv &= \left(m - |dm|\right)\left(v + dv\right) + |dm|\left(v - e\right) && \textit{where} \quad e \equiv \text{exhaust speed, relative to rocket} \\
&= mv + v\,dm + m\,dv - v\,dm + e\,dm && dm < 0, \text{ and discarding 2nd order terms} \\
\cancel{mv} &= \cancel{mv} + \cancel{v\,dm} + m\,dv - \cancel{v\,dm} + e\,dm \\
m\,dv &= -e\,dm \quad \text{or} \quad dv = -e\frac{dm}{m}, && dm < 0.
\end{aligned}$$

We take the exhaust speed *e* to be positive, even though it is directed leftward. The last line says the simple result we used above: the increase in momentum of the rocket (on the LHS) equals the (positive) decrease in momentum of the exhaust element (on the RHS).]

After a *finite* mass of exhaust has been ejected, what is the rocket's change in speed? We separated the variables *m* and *v*, so we can now simply integrate both sides:

$$\begin{aligned}
&\int dv = -e\int_{m_0}^{m_f} \frac{dm}{m}, \qquad \textit{where} \quad m_0 \equiv \textit{initial mass}, m_f \equiv \textit{final mass} \\
&\Delta v = -e\ln\left(m_f / m_0\right) = e\ln\left(m_0 / m_f\right)
\end{aligned} \tag{2.3}$$

Notice that:

In the absence of external forces, time is not a factor.

Δv depends only on throwing mass out in tiny increments at a fixed speed *e* (relative to the rocket).

However, it may be useful to know the rocket speed as a function of time. If the rate of fuel consumption is a constant $k > 0$, we can compute speed vs. time:

$$\frac{dm}{dt} \equiv -k < 0 \qquad \Rightarrow \qquad m(t) = m(0) - kt, \qquad \Delta v = v(t) - v(0) = e\ln\frac{m(0)}{m(0) - kt}$$

Recall that $m(t)$ is the mass of the rocket + fuel, so when the fuel runs out, $m(t)$ and $v(t)$ become constant. The denominator above (= mass remaining) is always positive.

Introducing the Relativistic Rocket

We now derive a similar relation between speed and remaining mass, for relativistic rockets and exhaust speeds. We use the same principle of conservation of momentum, but with two twists: (1) relativistic dynamics, and (2) including the mass of energy. First off, note that the speed of the exhaust mass in the observer frame is not $(v - e)$, because we must use the relativistic velocity addition relation:

$$w \equiv \text{exhaust speed in observer frame} = \frac{v - e}{1 - ve / c^2}, \qquad v, e \equiv \text{speed in observer frame}. \tag{2.4}$$

We will return to this at the end when we transform from the rocket frame to the observer frame.

Next, we must carefully define "mass:"

The mass of the rocket includes its rest mass, the fuel's rest mass,
and the mass of the potential energy stored in the fuel.

This total mass is the true rest-mass of the rocket (with fuel); however, I will use the term **mass-energy** for now [the term "inertia" seems more appropriate, but it might be confused with "momentum"]:

$m \equiv$ mass-energy of the rocket
= rest-mass of rocket + rest-mass of fuel + mass of fuel's energy

When the fuel is consumed to eject some exhaust, the exhaust has some rest-mass. In addition, the mass of the formerly potential energy used to expel the exhaust is also removed from the rocket, and carried off as *kinetic* energy of the exhaust. We choose to compute the masses from energies in the rocket frame, where they are simplest. Since mass is a (Lorentz) scalar, it can be computed in any frame, and is valid in every frame:

$$dm_{rocket} \equiv dm = -\left(dm_{exhaust} + \frac{PE_{lost}}{c^2} \right) < 0, \qquad PE \text{ infinitesimal} .$$

In an ideal engine, there would be no random heat-loss, and the potential energy lost equals the kinetic energy of the exhaust *measured in the rocket frame!* Equivalently, the *total* mass-energy lost by the rocket equals the *total* mass-energy of the exhaust:

$$\left| dm_{rocket} \right| c^2 = \gamma_e dm_{exhaust} c^2 \qquad \Rightarrow \qquad \left| dm_{rocket} \right| = \gamma_e dm_{exhaust}$$

$$\text{or} \qquad dm_{exhaust} = \left| dm_{rocket} \right| / \gamma_e \qquad \text{(ideal engine)} .$$

The *rest*-mass of the exhaust is *less than* the mass-energy lost by the rocket, because the PE of the fuel is counted as rest-mass in the rocket, but counted as kinetic energy in the exhaust. The rocket equation we seek is to be in terms of the mass-energy of the rocket, m, so we must eliminate $dm_{exhaust}$ from our equations, in favor of dm. But first, ...

Imperfect Efficiency

In any real engine, inefficiencies waste some potential energy that is not delivered as kinetic energy of the exhaust. This wasted energy further reduces the mass-energy of the rocket, but contributes nothing to its momentum. Since it adds little complexity to the problem, and is useful later, we now introduce efficiency into our analysis. On a first reading, you can just set $\eta = 1$, and skip this section.

As the rocket consumes its energy store, the energy goes into 4 places: the mass of the exhaust, the KE of the exhaust, the waste of inefficiency, and the increased KE of the rocket. This last bit, the increased KE of the rocket (in the rocket frame) does not contribute to the energy balance, because it is 2nd order in the differential speed, dv. We can show this with NR mechanics, because the rocket speed is zero in its own frame of reference:

$$KE \equiv T = \frac{1}{2} mv^2, \qquad dT = mv\, dv = 0 \qquad \text{because } v = 0 \text{ in the rocket frame} .$$

Now we account for conservation of energy: first, note that the $KE_{exhaust}$ is the PE lost times the efficiency $\eta < 1$:

$$KE_{exhaust} = \eta PE_{lost} \qquad \textit{where} \quad \eta < 1 \text{ is a given constant; smaller } \eta \Rightarrow \text{worse efficiency} .$$

Also, the relativistic KE of any mass is $(\gamma - 1)mc^2$. Therefore, still in the rocket frame:

$$KE_{exhaust} = (\gamma_e - 1)\, dm_{exhaust} c^2 = \eta PE_{lost}, \qquad KE, PE \text{ infinitesimal} .$$

$$\left| dm_{rocket} \right| = dm_{exhaust} + \frac{PE_{lost}}{c^2} = dm_{exhaust} + \frac{\gamma_e - 1}{\eta} dm_{exhaust} = dm_{exhaust} \left(1 + \frac{\gamma_e - 1}{\eta} \right)$$

Again, the increase in rocket KE is 2nd order, and therefore negligible *in this equation*, even though the increase in rocket KE is the ultimate desired result.

For brevity, we define a constant:

$$x \equiv \left(1 + \frac{\gamma_e - 1}{\eta} \right)^{-1} < 1 .$$

Then:

$$dm_{exhaust} = x \left| dm_{rocket} \right| \equiv x \left| dm \right| \qquad \textit{where} \quad x \text{ is a constant; smaller } x \Rightarrow \text{worse efficiency} . \tag{2.5}$$

For perfect efficiency, $\eta = 1$, and $x = 1/\gamma_e$.

Relativistic Rocket Equation in the Observer Frame

Finally, we write conservation of momentum *in the observer frame* for an infinitesimal mass-energy reduction of the rocket, including the inefficient PE reduction in the rocket. Conservation of momentum is:

$$p_{before} = p_{after} \quad \text{or} \quad \underbrace{\sum_{\text{masses}}}_{\text{before}} \gamma_v mv = \underbrace{\sum_{\text{masses}}}_{\text{after}} \gamma_v mv \quad \text{(observer frame)}. \tag{2.6}$$

Note that the masses are *rest* masses, and we must not include the "mass" of the particles' kinetic energy; any consequence of the mass of KE is already included in the formula for momentum. However, the mass of the fuel's PE *is* rest mass, and *must be* included in the momentum.

The momentum of the rocket just before it ejects a mass of exhaust is simply $\gamma_v mv$. Recall we defined w in eq. (2.4) as the exhaust speed in the observer frame, so then:

$$\gamma_v mv = \gamma_{v+dv}\left(m+dm\right)\left(v+dv\right) + \gamma_w \underbrace{(-x\, dm)}_{dm_{exhaust}} w \qquad \textit{where} \quad dm < 0, \ \ w \equiv \text{exhaust speed}$$

$$= \left(\gamma_v + d\gamma\right)\left(mv + v\,dm + m\,dv\right) - \gamma_w wx\,dm$$

$$0 = \gamma_v v\,dm + \gamma_v m\,dv + d\gamma mv - \gamma_w wx\,dm \qquad \text{Use:} \quad d\gamma = \gamma_v^3 \frac{v}{c^2} dv$$

$$0 = \gamma_v v\,dm + \gamma_v m\,dv + \gamma_v{}^3 \frac{v}{c^2} mv\,dv - \gamma_w wx\,dm$$

Our problem involves the rocket variables m and v, and given constants e and η (and x). Therefore, we would like to eliminate $\gamma_w w$ in favor of v and e. We make an aside to compute $\gamma_w w$ in those quantities, using relativistic addition of velocities (still in the observer frame):

$$w = \frac{v-e}{1-ve/c^2},$$

$$\gamma_w = \left(1 - \frac{w^2}{c^2}\right)^{-1/2} = \left(1 - \frac{(v-e)^2}{c^2\left(1-ve/c^2\right)^2}\right)^{-1/2} = \left(\frac{c^2\left(1-ve/c^2\right)^2 - (v-e)^2}{c^2\left(1-ve/c^2\right)^2}\right)^{-1/2}$$

$$\gamma_w w = \left[\frac{c^2\left(1-ve/c^2\right)^2 - (v-e)^2}{c^2\cancel{\left(1-ve/c^2\right)^2}}\right]^{-1/2} \left(\frac{v-e}{\cancel{1-\frac{ve}{c^2}}}\right), \qquad \text{using:} \quad \left[\frac{1}{\left(1-ve/c^2\right)^2}\right]^{-1/2} \left(\frac{1}{1-\frac{ve}{c^2}}\right) = 1.$$

Then:

$$\gamma_w w = \left[\frac{c^2 - 2ve + (ve)^2/c^2 - v^2 + 2ve - e^2}{c^2}\right]^{-1/2}(v-e)$$
$$= \left[\frac{c^2 + (ve)^2/c^2 - v^2 - e^2}{c^2}\right]^{-1/2}(v-e)$$
$$= \left[1 + (ve)^2/c^4 - v^2/c^2 - e^2/c^2\right]^{-1/2}(v-e)$$
$$= \left[1 + \left(v^2/c^2\right)\left(e^2/c^2\right) - v^2/c^2 - e^2/c^2\right]^{-1/2}(v-e)$$
$$= \left[\left(1 - v^2/c^2\right)\left(1 - e^2/c^2\right)\right]^{-1/2}(v-e)$$
$$= \gamma_v \gamma_e (v-e)$$

Plugging into the conservation of momentum equation:

$$0 = \gamma_v v\, dm + \gamma_v m\, dv + \gamma_v{}^3 \frac{v}{c^2} mv\, dv - \gamma_v \gamma_e (v-e)\, x\, dm \qquad \text{Divide by } \gamma_v$$
$$0 = v\, dm + m\, dv + \gamma_v^2 \frac{v^2}{c^2} m\, dv - \gamma_e (v-e)\, x\, dm \qquad \text{Use:} \quad \gamma_v^2 \frac{v^2}{c^2} = \frac{v^2}{c^2 - v^2}$$
$$0 = \left(v - \gamma_e (v-e)\, x\right) dm + m\, dv \left(1 + \frac{v^2}{c^2 - v^2}\right)$$
$$0 = \left(v - \gamma_e (v-e)\, x\right) dm + m \gamma_v^2 dv$$

This differential equation separates. Recall that e, γ_e, and x are given constants:

$$-m\gamma_v^2 dv = \left(v - \gamma_e x v + \gamma_e x e\right) dm$$
$$-\gamma_v^2 \frac{dv}{(1 - \gamma_e x) v + \gamma_e x e} = \frac{dm}{m} \qquad \textit{where} \quad dm < 0. \tag{2.7}$$

Solving this would give the mass remaining as a function of speed, $m(v)$. Note that both sides are dimensionless. [The left side can probably be expanded by partial fractions, and integrated.]

In the limit of perfect efficiency, $x = 1/\gamma_e$, and all the γ_e drop out. We get:

$$-\gamma_v^2 \frac{dv}{e} = \frac{dm}{m} \qquad \textit{where} \quad dm < 0.$$

To write this in terms of rates, we divide by dt in the observer frame:

$$-\gamma_v^2 m(t) \frac{dv}{dt} = e \frac{dm}{dt} \qquad \textit{where} \quad dm < 0, \gamma_v = \gamma_v(t)\ .$$

While this is true in any frame, note that v, t, and therefore dm/dt are all frame-*dependent*. Usually, the rocket consumes fuel at a *constant* rate $dm/d\tau$, where time τ is measured *in the rocket frame*. The rate dm/dt, measured by an inertial observer, is time dilated, and *varies* with the rocket's speed.

Consistency check: In the low-speed limit, (2.7) should recover the non-relativistic (NR) rocket equation. This is true for arbitrary efficiency η, because the mass of PE_{lost} is insignificant at NR speeds. In this limit:

$$\gamma_e \to 1, \quad x \to 1, \quad \gamma_v \to 1 \quad \Rightarrow \qquad -\frac{dv}{e} = \frac{dm}{m},$$

which is the NR rocket equation (2.2) above.

Calculations on Rocket Exhaust Strategies

In what sized chunks should we expel exhaust? In the above analyses, we have assumed a fixed exhaust speed, and ignored practical questions such as "How much fuel should we put on a rocket?", and "What mass to energy ratio makes the best fuel?" As background for such questions, we now consider energy and expulsion strategies. In contrast to the above, the exhaust speed *e* is now variable and to-be-determined.

Consider a rocket with a fixed amount of fuel to expel, m_{fuel}, and a fixed energy to expel it, *E*. Should we expel it all at once, as a big block of mass m_{fuel} with energy *E*? What about dividing it up, and expelling smaller chunks, one at a time? Let's calculate the rocket momentum for each of three ways: (1) expelling the fuel all at once, (2) expelling half, then the other half, and (3) expelling continuously (as a real rocket).

Figure 2.4: Different strategies for expelling exhaust.

For simplicity, we compute non-relativistically. Assume the fuel weighs [sic] twice the empty rocket: $m_{fuel} = 2m_{empty}$. *The nature of our question is now very different:* for fixed *energy* (not exhaust speed), how fast does the rocket go? We must find the rocket speed, *v*, after consuming all the fuel.

(1) Expelling all at once: With exhaust speed unknown, we must satisfy conservation of *both* momentum and energy (in contrast to being given a fixed *e*, where both conservation laws give equivalent information). As before, we take the exhaust speed *e* as positive, even though it's going leftward. Conservation of momentum gives:

$$m_{fuel} e = m_{empty} v, \qquad where \quad e \equiv \text{exhaust speed}. \qquad \text{Use: } m_{empty} = \frac{1}{2} m_{fuel}$$

$$m_{fuel} e = \frac{1}{2} m_{fuel} v \quad \Rightarrow \quad e = v/2$$

This is intuitive: consider the rocket starts at rest, then ejects its exhaust. The magnitude of the leftward momentum of the exhaust equals the rightward momentum of the rocket.

Now conservation of energy:

$$\frac{1}{2} m_{fuel} e^2 + \frac{1}{2} m_{empty} v^2 = E$$

$$\frac{1}{2} m_{fuel} \frac{v^2}{4} + \frac{1}{2}\left(\frac{1}{2} m_{fuel}\right) v^2 = m_{fuel} \frac{v^2}{8} + m_{fuel} \frac{v^2}{4} = E$$

$$\Rightarrow \qquad v = \sqrt{\frac{8E}{3m_{fuel}}} = 1.63\sqrt{\frac{E}{m_{fuel}}}$$

(2) Expelling in two steps: Step 1: After using half the energy to expel half the fuel, conservation of momentum says:

$$\frac{1}{2} m_{fuel} e = \left(\frac{1}{2} m_{fuel} + m_{empty}\right) v, \qquad \Rightarrow \qquad e = 2v \qquad where \quad e \equiv \text{exhaust speed}.$$

Conservation of energy says:

$$\frac{1}{2}\left(\frac{1}{2}m_{fuel}\right)e^2 + \frac{1}{2}\left(\frac{1}{2}m_{fuel} + m_{empty}\right)v^2 = \frac{E}{2}$$

$$2m_{fuel}v^2 + m_{fuel}v^2 = E \qquad \Rightarrow \qquad v_1 = \sqrt{\frac{E}{3m_{fuel}}}$$

We now have left a rocket with fuel remaining = $m_{fuel}/2$, and energy remaining = $E/2$.

Ejecting the remaining fuel, we again satisfying conservation of momentum; the momentum lost by the exhaust is gained by the (now empty) rocket:

$$\frac{1}{2}m_{fuel}e = \frac{1}{2}m_{fuel}\Delta v, \qquad \Rightarrow \qquad e = \Delta v \qquad where \quad e \equiv \text{exhaust speed relative to rocket} .$$

The (positive) decrease in exhaust momentum (LHS) equals the increase in rocket momentum (RHS). We can simplify the conservation of energy equation by noting that energy is conserved in *every* frame (even non-relativistically), so we choose the rocket frame just before expelling the exhaust:

$$\frac{1}{2}\left(\frac{1}{2}m_{fuel}\right)e^2 + \frac{1}{2}m_{empty}(\Delta v)^2 = \frac{E}{2}$$

$$\frac{1}{2}m_{fuel}(\Delta v)^2 + \frac{1}{2}m_{fuel}(\Delta v)^2 = E \quad \Rightarrow \qquad \Delta v = \sqrt{\frac{E}{m_{fuel}}}$$

$$v_{final} = v_1 + \Delta v = \sqrt{E/m_{fuel}}\left(\sqrt{1/3}+1\right) = 1.58\sqrt{E/m_{fuel}}$$

This is slightly less than the speed from expelling the fuel all at once. This makes sense: our second chunk is moving slower to the left than the first chunk (or even slightly to the right), because the rocket was moving to the right when it expelled the second chunk.

In general, for expelling a chunk of fraction Δf of the total fuel, leaving fraction f remaining, we can compute Δv (and e). Conservation of momentum says:

$$\Delta f\ m_{fuel}e = \left(f\ m_{fuel} + m_{empty}\right)\Delta v \quad \Rightarrow \qquad e = \frac{f\ m_{fuel} + m_{empty}}{\Delta f\ m_{fuel}}\Delta v .$$

Then conservation of energy:

$$\frac{1}{2}\Delta f\ m_{fuel}e^2 + \frac{1}{2}\left(f\ m_{fuel} + m_{empty}\right)(\Delta v)^2 = \Delta f\ E .$$

Substituting out e:

$$\Delta f\ m_{fuel}\left(\frac{f\ m_{fuel} + m_{empty}}{\Delta f\ m_{fuel}}\right)^2(\Delta v)^2 + \left(f\ m_{fuel} + m_{empty}\right)(\Delta v)^2 = 2\Delta f\ E$$

$$\Delta v = \sqrt{2\Delta f\ E}\left[\frac{\left(f\ m_{fuel} + m_{empty}\right)^2}{\Delta f\ m_{fuel}} + f\ m_{fuel} + m_{empty}\right]^{-1/2}$$

Consistency check: Numerical integration of the above, breaking the fuel into > 100 chunks, recovers the result (2.3) already obtained for continuous expulsion, thus confirming consistency. [See Python code at end of section.]

(3) Expelling continuously: If we expel the exhaust continuously, in infinitesimal chunks, we recover the situation for which we derived the NR rocket equation (2.3), but now with energy E given, and exhaust speed e to be determined. Consider an infinitesimal amount of fuel exhausted, dm. Conservation of energy, in the rocket frame, gives:

$$\underbrace{\frac{1}{2}e^2\left|dm\right|}_{\text{kinetic energy}} = \underbrace{\frac{E}{m_{fuel}}\left|dm\right|}_{\Delta PE} \quad \Rightarrow \quad e = \sqrt{2E / m_{fuel}} \tag{2.8}$$

Thus, the exhaust speed (relative to the rocket) is constant, and given by the energy density of the fuel (E/m_{fuel}).

As shown earlier, the increase in kinetic energy of the rocket is 2nd order in the differentials, so does not consume any of the infinitesimal fuel energy (in the frame of the rocket). Therefore, in our current rocket example, after consuming all the fuel, eq. (2.3) gives:

$$v_{final} = e \ln \frac{m_{fuel} + m_{empty}}{m_{empty}} = e \ln 3 = 1.55\sqrt{E / m_{fuel}} \,. \tag{2.9}$$

We see that in general:

Expelling exhaust in bigger chunks yields slightly higher final speed.

The best strategy, if possible, is to expel all the fuel in one giant push. In practice, this is infeasible because a realistic system cannot convert all the energy of the fuel at one time. However, in this NR example, with 2/3 of the initial rocket mass being fuel, the continuous expulsion of exhaust costs only ~5% of the best possible final speed (1.55 vs. 1.63).

In general, (2.8) and (2.9) give the final speed of a continuously exhausting rocket in terms of m_{empty}, m_{fuel} and the energy to expel the fuel, E. Typically, the fuel contains the energy to expel it. Then E/m_{fuel} is constant. To make faster rockets, we increase the m_{fuel} to m_{empty} ratio. Eventually, m_{fuel} grows to be >> m_{empty}, but the final speed grows only logarithmically with m_{fuel}. This is very slow growth.

Python code to compute the expulsion of fuel in chunks:

```
""" Rocket.py: expel fuel in chunks
Usage: rocket.py rho m_fuel m_empty N_chunks
"""
import sys                # argv[]

fr = 1                    # fuel fraction remaining
rho = float(sys.argv[1])# energy density: E/mf
mf = float(sys.argv[2]) # initial mass of fuel
me = float(sys.argv[3]) # mass of empty rocket
n = int(sys.argv[4])      # number of chunks to expel

df = 1./n         # fractional change in fuel per chunk
v = 0             # current speed of rocket

for nc in range(1,n+1): # expel chunks 1..n
    fr -= df            # decrease fraction of fuel remaining
    dv = ((fr*mf + me)**2)/(df*mf) + fr*mf + me
    dv = 1/sqrt(dv)
    dv *= sqrt(2*df*rho*mf)
    v += dv             # accumulate current speed
    print dv, v         # delta-v, current speed
```

Ion Engines, and What Kind of Exhaust Should We Use?

One web source claims that the higher the exhaust speed, the more effective the rocket, and therefore you should use *light* as your exhaust, since it travels at the fastest possible speed. We now show that this is true only in the unrealistically ideal case of perfect engine efficiency.

So far, we have been considering engines such as existing fuel-burning engines, where burning provides the energy to expel the burnt fuel as exhaust. In other words, the fuel itself becomes the exhaust mass (after burning). Our analysis, though, can be more general, to work with engines where the energy source is completely separate from the exhaust mass, such as an ion engine.

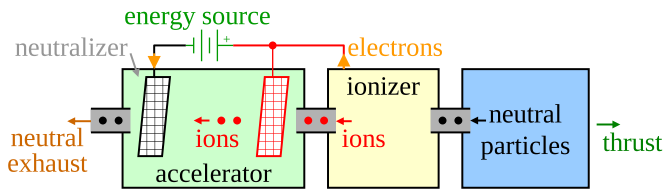


Figure 2.5: Ion engine: electric energies are much higher than thermal energies.

Ion engines work as in Figure 2.5: a mass of atoms is ionized; then electric energy accelerates the ions. Finally, as the ions near the exhaust port, they are neutralized with their own electrons. They thus leave the engine at high speed, propelling the engine. The advantage is that electric energies of ~10,000 eV are easily obtained, whereas thermal energies at 1000 K are ~0.1 eV.]

Again, consider a relativistic engine with a fixed energy supply (and its associated mass), and a fixed exhaust mass to expel. How fast should we expel the exhaust? Recall that our goal is to get the most momentum for our fuel energy. In other words, we seek to maximize the momentum-to-energy ratio, by varying the fuel energy per unit exhaust mass.

First, to show the principle, let's assume perfect efficiency. The exhaust particle speed e is now a variable, and is determined by its mass, and the available fuel energy. Using relativistic dynamics, in the rocket frame:

$$dE_{exhaust} = -dE_{rocket} \qquad \Rightarrow \qquad \gamma_e dm_{exhaust} c^2 = -c^2\ dm$$

$$\textit{where } \text{(as before)} \qquad dm \equiv dm_{rocket} = -\left(dm_{exhaust} + \frac{PE_{lost}}{c^2} \right) < 0 .$$

Then the exhaust mass and momentum are:

$$dm_{exhaust} = |dm| / \gamma_e, \qquad dp_{exhaust} = \gamma_e dm_{exhaust} e = e\,|dm| .$$

The momentum is e times the rocket mass-energy reduction. Therefore, the maximum momentum per fuel-mass occurs when e is maximized. Thus we choose a minimum exhaust particle mass, which is zero for electromagnetic radiation, and e is maximized at the speed of light. Thus, for perfect energy conversion efficiency, the most efficient engine is not at ion engine, but one where the fuel annihilates completely into radiation: a matter/anti-matter combination (like Star Trek).

Now, as before, we introduce the inefficiency of a more plausible engine. We derived earlier the relativistic relationship between the rocket mass-loss and the ejected exhaust mass (2.5).

$$dm_{exhaust} = x\,|dm| \qquad \textit{where} \quad x \equiv \left(1 + \frac{\gamma_e - 1}{\eta} \right)^{-1} < 1; \qquad \text{smaller } x \to \text{worse efficiency} .$$

η is now a given constant, but e is variable, and therefore so are γ_e and x. The exhaust momentum is:

$$dp_{exhaust} = \gamma_e dm_{exhaust} e = \gamma_e e x\,|dm| \qquad \textit{where} \quad dm \equiv dm_{rocket} < 0 .$$

(This reduces to the above when $\eta = 1$.) Therefore, we must maximize the product $\gamma_e e x$, determined entirely by e:

$$\gamma_e e x = \gamma_e e \left(1 + \frac{\gamma_e - 1}{\eta} \right)^{-1} = \frac{\eta \gamma_e}{\eta + \gamma_e - 1} e$$

I haven't got the energy right now (pun intended) to analytically find the maximum, but numerical plots of $\gamma_e e x$ (Figure 2.6) show that for a low-to-realistic $\eta = 0.2$, the optimum exhaust speed is ~0.6c, somewhat smaller than c. Higher η prefer even higher exhaust speed. However, even an ion engine with 10 keV hydrogen exhaust reaches only ~0.005c, so in practice, the faster the better.

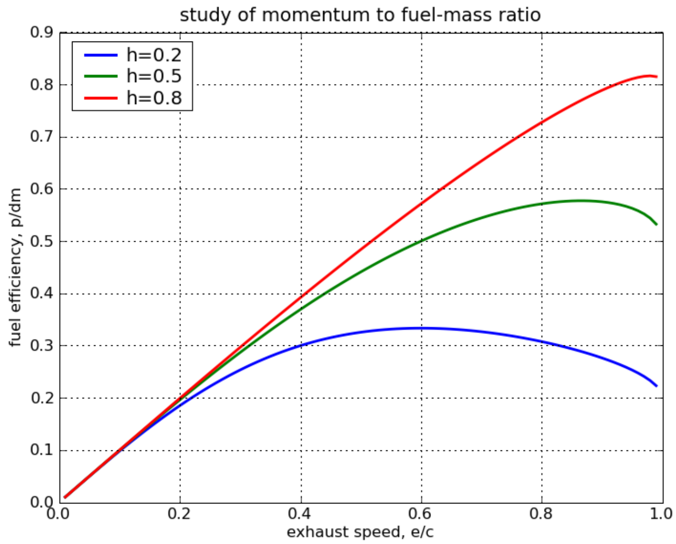


Figure 2.6: Some efficiencies of fuel-to-energy conversion in a relativistic rocket.

Electromagnetic Exhaust

TBS. Exhausting light. Never run out of photons. Efficiency is still pretty good. RTG thermal emission exhaust. Not PV, solar sails. Tacking "upwind"?

Viscosity of light. CMB dipole thrust.

Common Mistakes

When ejecting a mass *dm*, be careful that the *total* energy of the rocket (rest + kinetic, in the observer frame) decreases by the total energy of the exhaust $\gamma_w dm_{exhaust}$, also in the observer frame. (In fact, this energy equality is true in *any* frame.) Note that the rocket energy lost (in the observer frame) is *not* $\gamma_e dm_{exhaust}$; depending on *v*, it is sometimes greater than, and sometimes less than, $\gamma_e dm_{exhaust}$. As we noted below (2.6), conservation of momentum already *includes* the "mass" of kinetic energy (thus treating kinetic energy differently than other forms of energy).

The rocket's rest-mass loss (including PE of the fuel) is greater than the *rest*-mass of the ejected fuel (except at the instant $v = e$). The change in rest mass of the rocket can be computed from the exhaust kinetic energy *measured in the frame of the rocket*, as we did above. Alternatively, it can be computed in the frame of the rocket, if the change in rocket KE is also included.

Spacetime Diagrams

Spacetime diagrams are very helpful in visualizing relationships,. They are essential for some concepts, but few references describe them in detail. We first look at the characteristics of a spacetime diagram in a single reference frame. Then we look at how two different reference frames appear on a single diagram. We do not *derive* anything; we simply present the results, in hopes that you will understand the meaning. Then you can look up the derivation in a standard SR text.

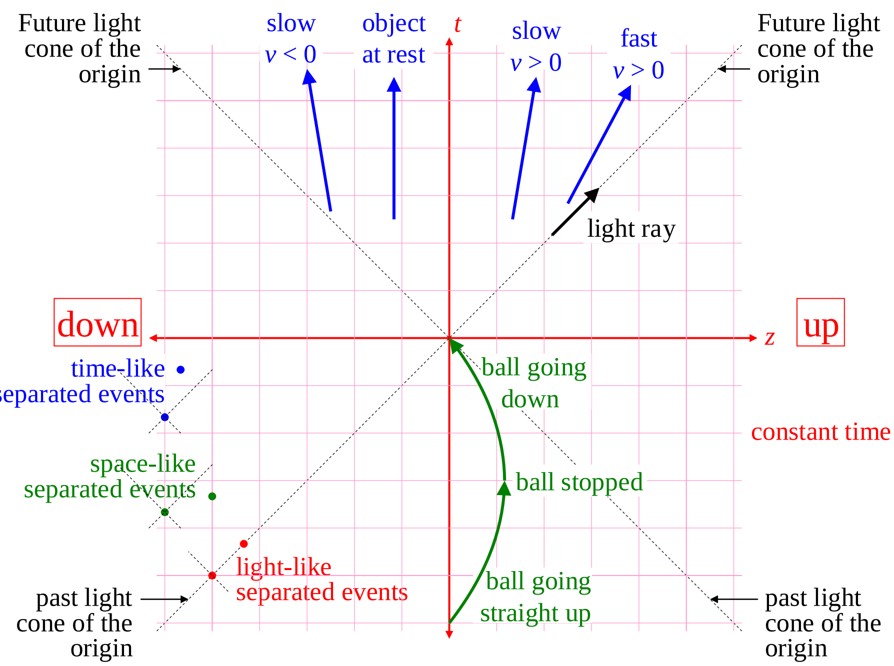


Figure 2.7: Spacetime diagram, with examples of motion drawn on it. Positive z is upward in space. The ball's speed is greatly exaggerated for illustration.

Usually, we have room for only one space coordinate, and usually we draw it as the abscissa (horizontal axis). In Figure 2.7, we choose z as the vertical direction. An object at rest traces a world line straight up on the diagram: it moves only through time. An object rising slowly upward (in space) follows a path slightly tilted to the right. An object rising quickly is more tilted to the right. A light ray going up follows a 45° line to the right (because we measure time in meters, and c = dimensionless 1). If we throw a ball straight up into the air, it makes a parabola first moving to the right (on the diagram, which is *up* in space), then going left (on the diagram, which is falling back *down* in space).

Light cones: At any point on the spacetime diagram (not just the origin), we can draw a light-cone: lines at 45°. These are the trajectories of light rays ($v = 1$). Since massive objects move slower than light, massive objects must follow world lines inside the forward light cone *from every point on the world line*; i.e. the magnitude of the slope of the world line must always be > 1. Similarly, every massive object must have come *from* a point within the past light cone.

Time-like separation: Consider two events, at different times, and possibly different places. If the second event occurs inside the (future) light-cone of the first, the events are **time-like** separated. This means that *all* inertial observers see the events occurring in the same order, and the first may be a *cause* of the second. The events are said to be **causally connected**. Note that for time-like separated events, there always exists an inertial reference frame in which the events occur at the same place.

Space-like separation: Consider two events, at different places, and possibly different times. (If they occur at the same time in some frame, pick one arbitrarily, and call it "first.") If the second event occurs *outside* the future light cone of the first, the events are **space-like** separated. This means that *all* inertial observers see the events occurring at different places, and that not even light can travel fast enough for one event to influence the other. The events are said to be **causally-disconnected**. Note that for space-like separated events, there always exists an inertial reference frame in which the event occur at the same time.

Light-like separation: At the boundary between time-like separation and space-like separation is **light-like** separation. Consider two events, at different places and times. If a light ray from the first event could just reach the second event, the events are light-like separated. In principle, this could allow them to be causally connected, though in reality, other inevitable delays in causation make them causally disconnected. In practice, the causality of light-like separations is irrelevant.

Oblique Coordinates

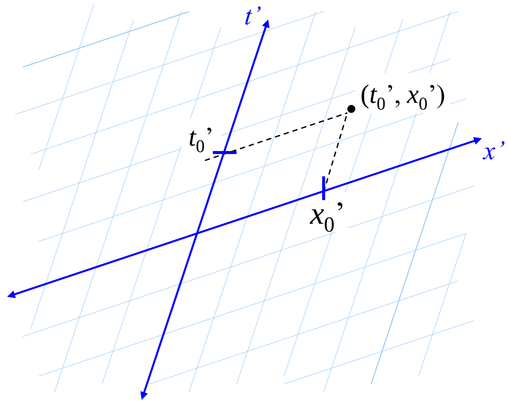


Figure 2.8: Oblique coordinates are almost as good as orthogonal ones: the coordinates of any point are still unique.

We are used to parametrizing space with orthogonal (cartesian) coordinates. However, it is often convenient to use oblique (meaning: non-orthogonal) coordinates, as shown in Figure 2.8. Each point in space is still given by a unique coordinate pair. The metric (tensor field) for such coordinates in a flat space is constant throughout the space. Spacetime diagrams often have one set of perpendicular coordinates, and another set of oblique coordinates.

Multiple Observers on the Spacetime Diagram

It is often helpful to consider how two different observers see the same set of events. For this, we draw a set of time and space axes for each observer. The angles of the axes derive from the Lorentz transformation [ref??]. Consider two observers, named Blue and Red. Blue moves to the right with respect to Red, along the *x* axis (Figure 2.9).

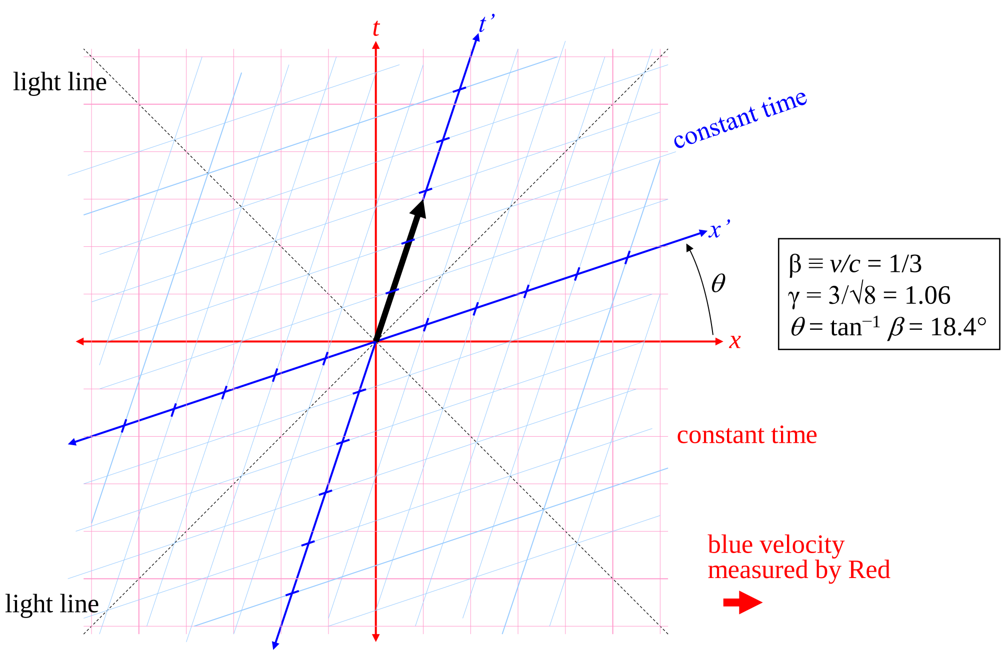


Figure 2.9: The blue observer moves to the right with respect to the red observer, along his own time axis.

We arbitrarily choose the red system to have its time and space axes orthogonal.

There is no physical significance to which frame is drawn with orthogonal spacetime axes.
This frame is not preferred in any way.

The blue lines label the same events in spacetime, but with time and space measured from Blue's frame. This requires the blue coordinate lines to be oblique (non-perpendicular). Each observer is stationary in his own frame, i.e. his trajectory follows his own time axis (e.g, Blue follows *x*' = 0, along the black arrow). In this example, the lines of constant time are horizontal for red, and tilted up to the right for blue. The light cone is the path of a light ray, which is the same for all observers. Blue's axes are "squished" toward the future light cone in the direction of his motion relative to the orthogonal (red) axes.

Constructing the Primed Axes

Constructing the *t*' axis is simple: it is the position of the primed (Blue) origin (0, 0)', as seen in the Red frame. However, this does not define the *scale* of time on the axis.

Construction of the x' Axis

This is a little trickier. We use the physical principle that the speed of light is the same for both Blue and Red. In the primed frame, the *x*' axis is the set of events that all occur at the same time *t*' = 0. To find the *x*' axis, we can find any two points on it. For illustration, we construct 3 events (reflections) to occur at *t*' = 0. Imagine Blue emits 3 pulses of light moving to the right (Figure 2.10), emitted from *x*' = 0 at *t*' = –3, *t*' = –2, and *t*' = –1 (in arbitrary units).

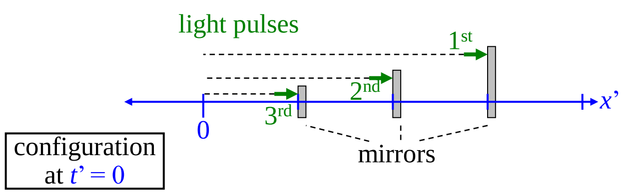


Figure 2.10: At *t*' = 0, light pulses and mirrors in the primed frame.

We have placed mirrors so that at *t*' = 0, all 3 pulses are simultaneously reflected back toward the origin. The pulses hit the origin in reverse order of their launch, at times *t*' = 1, *t*' = 2, and *t*' = 3. In the primed (blue) frame, the system looks like Figure 2.11a.

We now draw the same system in the unprimed (red) frame. The cleverness of the reflection is now apparent: we don't need to know the *scale* of the *t*' axis, because we know the pulses take just as long (in the primed frame) to return to the primed origin as to get to the mirrors (Figure 2.11b). We mark off equal time units (of arbitrary size) along *t*'. In the unprimed (red) frame (and in any frame), light follows 45° lines, so we construct the forward progress of each light pulse as 45° lines from their (spacetime) points of origin. We know they eventually arrive back at the *t*' axis (which is *x*' = 0) at times *t*' = 1, 2, and 3. We draw the 45° light-lines backwards from *t*' = 1, 2, and 3. All of them intersect their outgoing light-lines at *t*' = 0, somewhere on the *x*' axis. Thus the three intersections points lie along the *x*' axis.

Note that we can pick any two *arbitrary* launch times before *t*' = 0, and still construct the spacetime diagram similarly to Figure 2.11.

The key piece of physics in this construction is that the speed of light is the same in the red and blue frames. Light lines are 45° on the spacetime diagram for both frames.

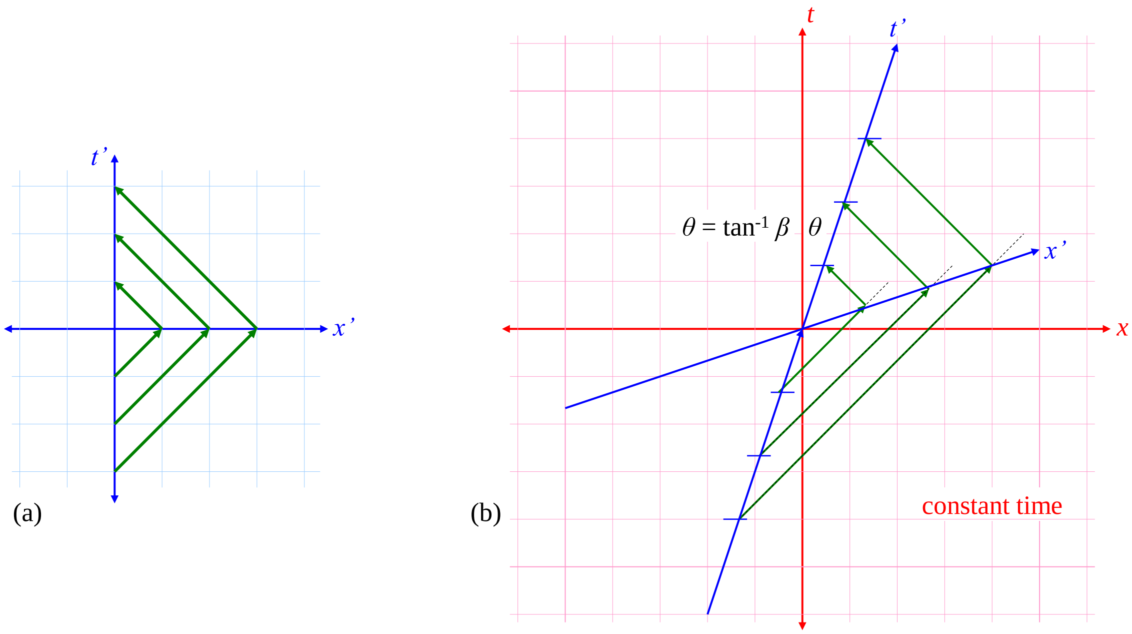


Figure 2.11: Construction of the *x*'-axis, on a diagram where (*t*, *x*) are orthogonal (i.e., the "lab" coordinates). (a) Pulses in the primed frame. (b) Spacetime diagram with both observers' coordinates (blue and red).

Scaling of the tilted axes

Note that a unit of time or space on the oblique (non-perpendicular) axes is drawn larger on the diagram than a unit of time or space on the perpendicular axes. We can visualize this with a "calibration hyperbola" (Figure 2.12). We've chosen a faster relative speed for Blue than before, so the blue axes are tilted more than before.

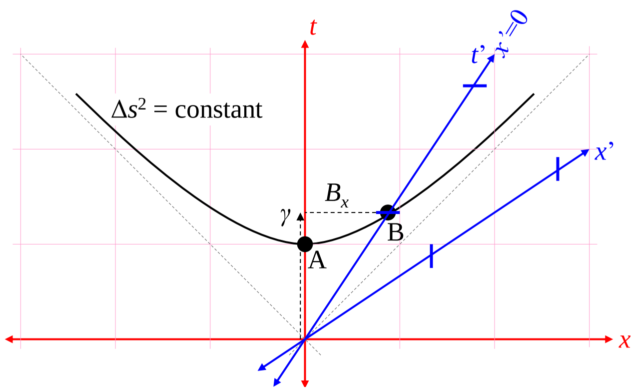


Figure 2.12: A "calibration hyperbola" allows us to visualize the γ factor.

In the unprimed coordinates, we can construct the locus of events with equal interval separation from the origin:

$$\Delta s^2 = -t^2 + x^2 = const = 1 \text{ (say)} .$$

This is the equation for a hyperbola, as shown. Still in the unprimed frame, A = (t = 1, x = 0). Since the interval is a Lorentz invariant (all observers compute the same interval), the hyperbola also satisfies $(\Delta s')^2$ = 1 in the primed frame. We choose the event B to be on the *t*' axis (*x*' = 0) where it crosses the hyperbola. Then:

$$\Delta s^2 = -t'^2 + x'^2 \quad \Rightarrow \quad t' = 1, \quad B = (t' = 1,\ x' = 0)\ .$$

Therefore, one unit of time in the primed frame is measured as *longer* than one unit by an unprimed observer, by the factor γ (as shown). We can compute γ from the geometry at B (where t' = 1), and the given relative speed β:

$$\beta = \frac{\Delta x}{\Delta t} = \frac{B_x}{\gamma} \quad \Rightarrow \quad B_x = \beta\gamma, \qquad \text{and}$$

$$\Delta s^2 = 1 = B_x{}^2 - \gamma^2 = \beta^2\gamma^2 - \gamma^2 = \left(\beta^2 - 1\right)\gamma^2 \Rightarrow \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Figure 2.13 shows closer looks at two observers on a single spacetime diagram, for two different relative speeds. Figure 2.13b is easier to see. Note that one blue second (OB) measures as *more* than a second to Red (time dilation), and a 1 light-second blue rod (OD) measures *shorter* than a light-second to Red (length contraction). In perfect symmetry, a red second (OA) also measures *more* than a second to Blue (follow the blue dashed line from A to the t' axis), and 1 red light-second (OC) measures *shorter* than a light-second to Blue. Other observers would have other length scales, with a unit being drawn longer for those closer to the light-cone (a 45° line).

> The Red and Blue views of each other are *symmetric*, not *reciprocal*. That is, Red measures Blue's clock running slowly, and by symmetry, Blue *also* measures Red's clock running slowly.

Though Red measures a blue clock as slow, Blue does *not* measure Red's clock running fast (which would be reciprocal)!

Later, in General Relativity, we will find relationships that are neither symmetric nor reciprocal (e.g., the twin paradox).

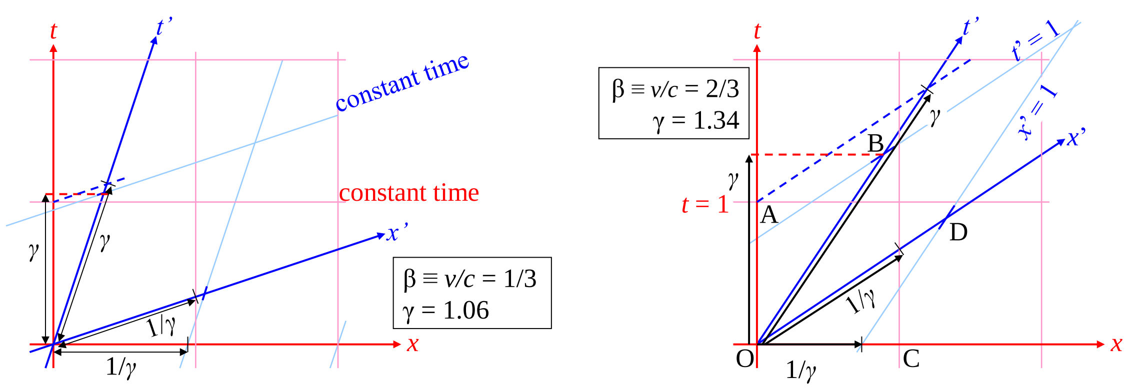


Figure 2.13 Detail of time dilation and length contraction at (a) β = 1/3 (barely visible), and (b) β = 2/3.

Again, there is nothing special about the frame in which the time and space axes are perpendicular. We can choose to draw any inertial frame this way, and all other frames' axes are determined by their motion relative to that frame. We can just as well draw the spacetime diagram with the blue axes perpendicular (Figure 2.14).

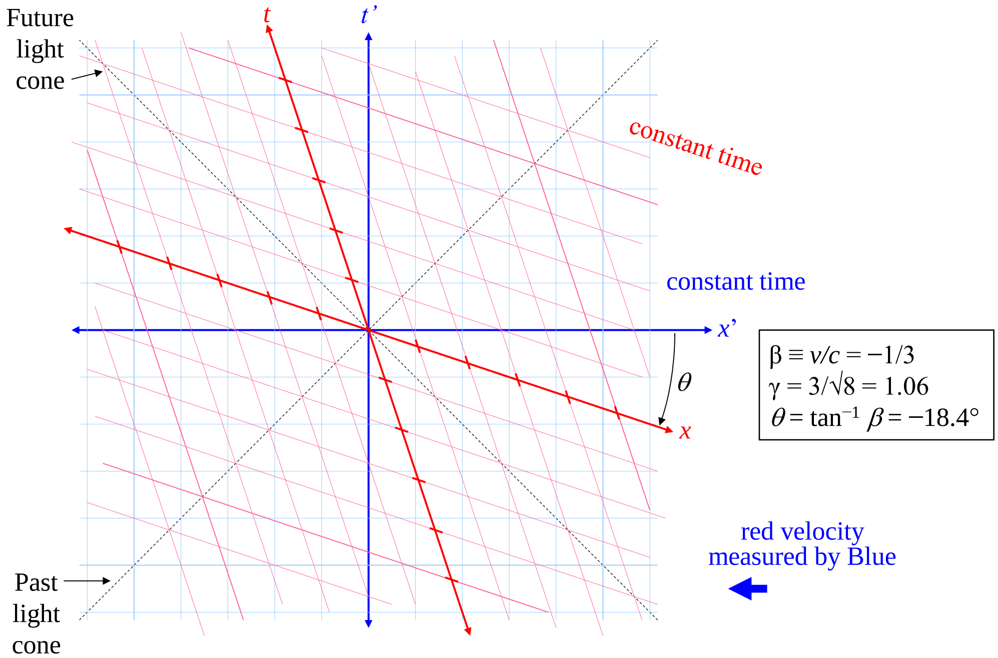


Figure 2.14 Equivalent spacetime diagram to Figure 2.9: the red observer moves to the left with respect to the blue observer.

Dinner at Alpha Centauri

Alpha Centauri is about 4 light-years away. Can I get there in time for dinner? Yes! Let us measure this from two different frames: my frame moving toward Alpha Centauri, and the earth ("stationary," or "lab") frame.

First, if I travel near the speed of light, the distance to Alpha Centauri shrinks. Then Alpha Centauri is much closer than 4 light-years. If I go fast enough, I can make it only 4 light-hours away. I'm traveling at nearly the speed of light, so I'll get there in four hours, in time for my dinner.

On earth, they measure me traveling at nearly c, and it takes me 4 years to get there. However, they also measure that my clock runs slowly, so it only advances 4 hours during the trip. At the end, all observers agree that my clock has advanced only 4 hours.

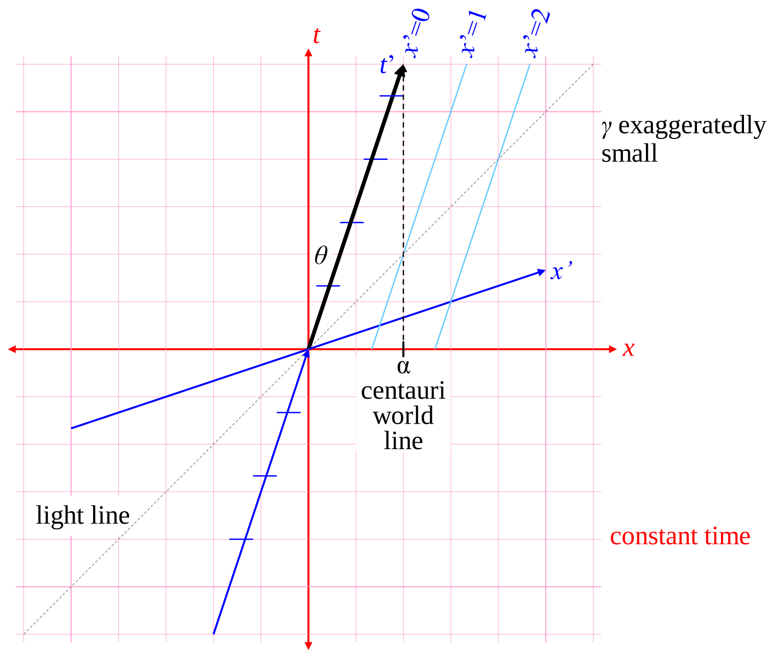


Figure 2.15 Two views of the trip to Alpha Centauri. In the red frame, the trip takes 6 time units. In the blue frame, it takes only about 4.5.

But how is it that all observers measure the earth clocks as having elapsed 4 years? Let us see.

Forward to the Future

Back to orthogonal Red axes: consider how Blue (me, the traveler) measures a set of red clocks, which are synchronized in the red frame (but not in the blue frame), Figure 2.16. Say the blue observer compares the red clock at the upper right with the 2nd from left on the lower row, at the same time in his frame (along the blue line labeled “constant time”). He measures that the upper right clock, which he is approaching, reads a later time.

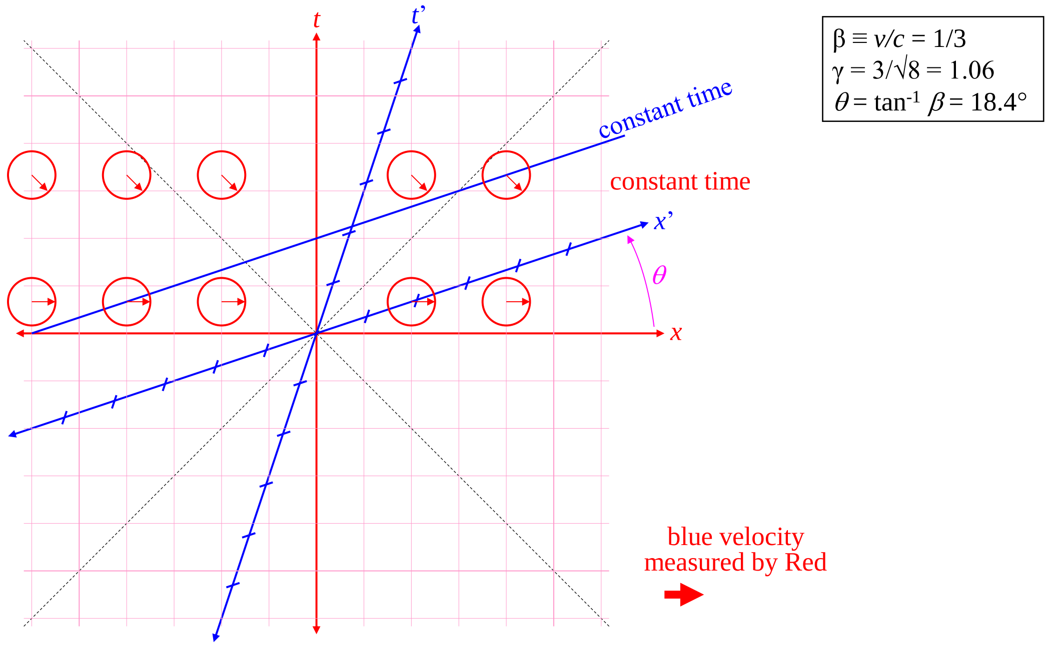


Figure 2.16 The 5 red clocks are synchronized in the red frame. They read different times in the blue frame (moving with respect to the red frame). The red clocks that Blue is approaching show later times than those receding.

When an inertial observer (Blue) looks at another inertial set of clocks (Red), they read *different times*, but they all run at the *same rate*, slowed by the factor $1/\gamma$. Clocks he is approaching read *later* than clocks that are receding.

In other words, "clocks ahead read ahead; clocks behind read behind."

Thus, at the beginning of my trip to Alpha-Centauri, I measure that the red clock (already *at* Alpha-Centauri) reads almost 4 years later. It advances a negligible amount (4 hours divided by a huge γ) during my trip, so when I get there, the clock still reads essentially 4 years later than mine. Thus, all observers agree that the earth clocks advanced 4 years during the trip.

If Blue moves even faster to the right, his axes squish tighter to the light cone. The graph gets hard to read (Figure 2.17).

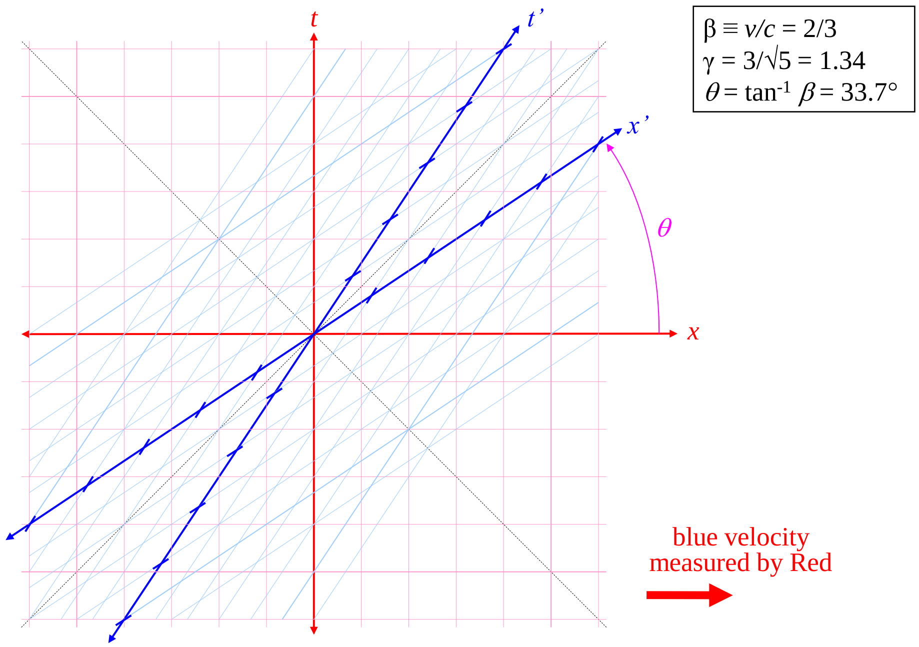


Figure 2.17 Blue moves to the right faster than the preceding diagrams. It gets hard to read.

Are Time and Space Equivalent?

Time and space are *not* on an equal footing. We've already seen this in the metric, because the coefficient of time is negative, and the coefficients of space are positive. Furthermore, for any observer, time always goes forward. Thus, there are "flows" of stuff forward in time in the spacetime continuum. For example, $T^{\mu\nu}$ is a flow of energy and momentum [MTW p130].

Time and 2D Space

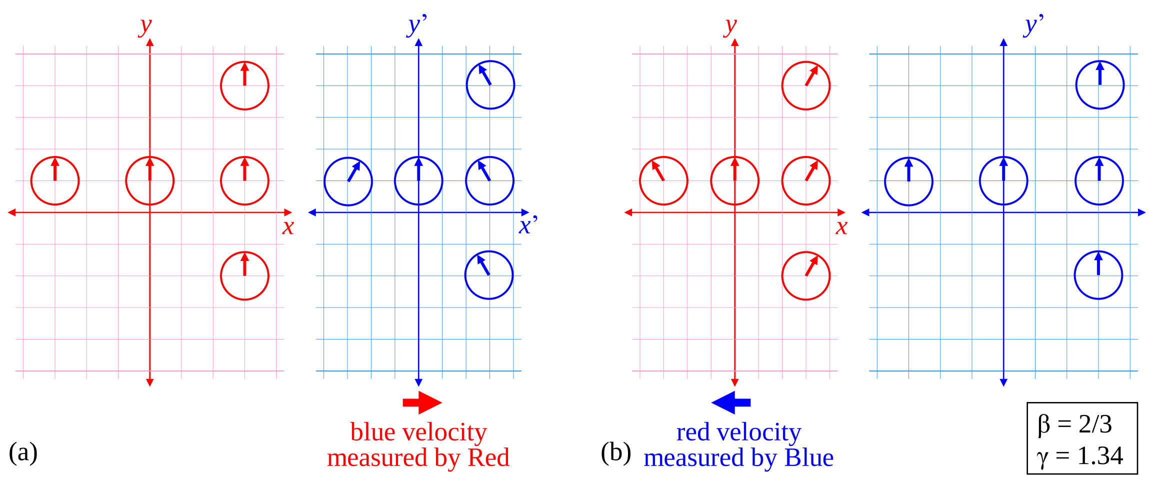


Figure 2.18 (a) Time and space measured by Red. (b) Time and space measured by Blue.

Figure 2.18 gives a qualitative snapshot of space and time as measured by Red and Blue. These diagrams are sometimes helpful in thinking about relativity questions. In the red frame, the *x*-*y* grid is square and the red clocks are synchronized, but the blue grid is contracted, and the blue clocks are skewed in the *x*-direction. Red does measure that all blue clocks at the *same x* position are synchronized.

Blue has the symmetric measurements: the blue grid is square, and the blue clocks are synchronized, but the red grid is contracted, and the red clocks are skewed. Blue measures that all red clocks at the same *x* position are synchronized.

Velocity Composition Experiment

Traditional relativistic velocity composition (addition) is a tired, old subject: derive Lorentz transformations, then use calculus to derive the velocity addition law. This obscures the physical meaning of the formulas. But a novel physical derivation (rigorous, but non-calculus) is an excellent exercise in important principles of Special Relativity, yielding physical insight. We derive the composition law here, from the physical considerations of time-dilation, length contraction, and clock relations applied to a thought experiment. This section requires only high-school algebra, except for the optional segment on "rapidity."

One Observer's Space Is Another's Time

Figure 2.16 illustrated graphically how a set of synchronized clocks looks to a moving observer. We now derive this with a physical experiment. Figure 2.19a shows some blue clocks, with a flash of light originating midway between two of them. The blue frame moves to the right with speed *v* relative to the red frame. (The velocity *v* is in red because it is the clock velocity *measured by* Red.) In Blue's frame, the pulses hit the clocks at the same time.

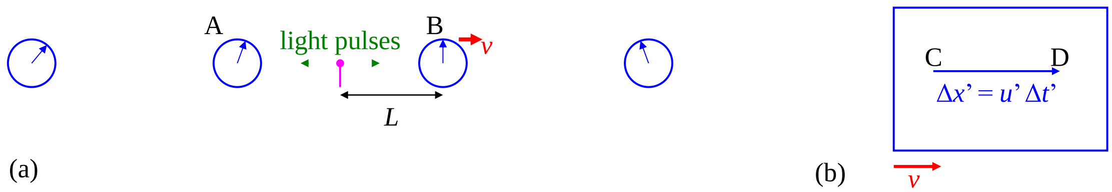


Figure 2.19: A light flashes from midway between two clocks. In Blue's frame (not shown), the pulses hit the clocks at the same time. (a) In Red's frame, the clocks are moving right, and the pulse hits clock A before B. (b) A point has velocity *u*' in Blue's frame, and Blue's frame has velocity *v* in Red's frame.

In Red's frame, the clocks are moving right, and the A pulse hits clock A before the B pulse hits clock B. It is crucial to understand that:

> All observers agree on the reading of a blue clock when the light pulse hits it.
> Even in General Relativity, all observers agree that coincident events are coincident.

Therefore, Red (like Blue) observes that the time on clock A when the pulse hits it is the same as the time on clock B when its pulse hits it. Since Red observes the A event before the B event, it must be that Red measures that clock A is set *ahead* of clock B (as shown), so clock A reads the time that clock B will read sometime later. As noted earlier, clocks that are physically ahead of us (coming toward us), read times that are ahead of clocks behind us (receding from us). And farther-ahead clocks read later times than nearer-ahead clocks.

Let's quantify this: In Figure 2.19a, in Red's frame, call the distance between the clocks 2*L*, and the time between emission and arrival at clock A Δt_A. We find Δt_A from the rate of closure of clock A onto the light pulse, which is $c + v$:

$$\Delta t_A = \frac{L}{c+v} .$$

Similarly, the rate of closure of the light pulse to clock B is $c - v$, so:

$$\Delta t_B = \frac{L}{c-v} .$$

Red sees that each clock reads the same time as the other when the respective pulse hits it. So clock A must be ahead of clock B (in Red's frame) by the difference in propagation time.

$$diff_{AB} = \Delta t_B - \Delta t_A = \frac{L}{c-v} - \frac{L}{c+v} = L\left(\frac{2v}{c^2 - v^2}\right) = 2L\frac{v}{c^2}\gamma_v{}^2 .$$

If Red sees clock A read exactly noon, this is how long Red has to wait for clock B to read noon. But Red measures all the blue clocks running slowly, so the time *on clock A's face* is ahead of B by less:

$$diff\,'_{AB} = \frac{diff_{AB}}{\gamma_v} = 2L\frac{v}{c^2}\gamma_v \qquad (L \text{ measured in red frame}) . \tag{2.10}$$

The clock time difference on their faces, as measured by Red, is proportional to their separation ($2L$), with proportionality constant $v\gamma_v/c^2$.

It can be argued that all the unexpected behavior of relativity can be attributed to the lack of simultaneity (non-synchronization) of clocks in a moving frame.

In sum: Considering different space points in the red frame is also considering different time points in the blue frame. When Red measures the whole set of blue clocks, he finds they all run at the same rate, albeit slowly, and they aren't synchronized. But Red knows how slowly they run, and by how much time they disagree, so Red can always convert between blue clock readings and his own (red) time. Red is now prepared to derive the velocity addition rules.

Three Pieces of Velocity Addition

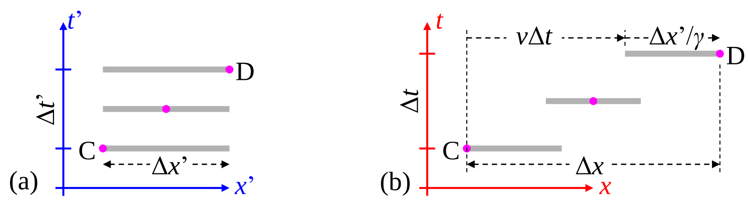


Figure 2.20: Snapshots of a point moving from one end of a stick to the other (a) in the blue frame; (b) in the red frame, where the stick is moving and contracted.

Parallel component: We first consider how velocities along the axis of frame motion compose. Later, we compute velocities perpendicular to the frame motion. To physically derive velocity addition, consider Figure 2.19b, and Figure 2.20a: a point moves (say, along a rod) with speed u' (in the blue frame) in the x-direction. How fast does it move in the red (unprimed) frame?

We must consider 3 factors in going from the blue frame to the red frame: space is different, time is different, and the blue clocks are not synchronized in the red frame. We are given u', $\Delta t'$ is arbitrary, and we trivially find $\Delta x' = u'\Delta t'$. Label the starting event C and the ending event D. Define time origins so that $t'_C = t_C = 0$; the blue time of event D is $\Delta t'$. From the previous section, we can find everything we need to compute u, the point's red velocity, in terms of v and the given primed quantities..

Imagine there are blue clocks at points C and D. In the red frame, if the blue clocks were synchronized, we could simply use length-contraction and time-dilation to take $\Delta x'$ and $\Delta t'$ to Δx and Δt, and find $u = \Delta x/\Delta t$. But alas, the blue clocks are *not* synchronized.

In Red's frame, looking at *two* moving clocks is hard, because they read different times. Looking at *one* moving clock is simple: it runs slower by γ. So first, Red converts the time of clock D's face at the end to the time on clock C's face (in the red frame) at the end, using (2.10) with $2L \to \Delta x'/\gamma$:

$$C_{elapsed,red} = D_{face} + diff\,'_{CD} = \Delta t' + \frac{\Delta x'}{\gamma_v}\frac{v\gamma_v}{c^2} = \Delta t' + u'\Delta t'\frac{v}{c^2} = \Delta t'\left(1 + \frac{u'v}{c^2}\right).$$

From simple time dilation (since red is now considering only the blue C clock):

$$\Delta t = \gamma_v C_{elapsed,red} = \gamma_v \Delta t' \left(1 + \frac{u'v}{c^2}\right). \tag{2.11}$$

The factor in parentheses corrects for the lack of simultaneity of the blue clocks.

We find the red spatial separation from the diagram:

$$\Delta x = \frac{\Delta x'}{\gamma_v} + v\Delta t\ . \tag{2.12}$$

(We might expect to replace Δt with $\gamma_v \Delta t'$, but the Δt will cancel below, anyway, from $u = \Delta x / \Delta t$.)

Finally, combining the last three equations:

$$u \equiv \frac{\Delta x}{\Delta t} = \frac{\Delta x' / \gamma_v}{\gamma_v \Delta t' \left(1 + \dfrac{u'v}{c^2}\right)} + v = \frac{u' \cancel{\Delta t'}}{\gamma_v{}^2 \cancel{\Delta t'} \left(1 + \dfrac{u'v}{c^2}\right)} + v\ .$$

This is a somewhat clumsy form, cleaned up with simple algebra. We also insert the notation specifying all this as being for the *x*-component of motion:

$$u_x = \frac{u'_x\left(1 - \cancel{v^2 / c^2}\right) + v\left(1 + \dfrac{u'_x \cancel{v}}{\cancel{c^2}}\right)}{1 + \dfrac{u'_x v}{c^2}} = \boxed{\frac{u'_x + v}{1 + u'_x v / c^2}}. \tag{2.13}$$

This form shows clearly that the composite velocity is just the sum of the velocities, corrected for the lack of simultaneity of the blue clocks with which u' is measured. The correction factor approaches ½ at high speeds.

Also, the rod itself plays no role; it's just a visual aid. The reasoning applies to all velocity additions, rod or not.

Figure 2.21 illustrates velocity addition on a spacetime diagram. If $u'_x < c$ and $v < c$, then the resulting velocity (dashed magenta) is also $< c$.

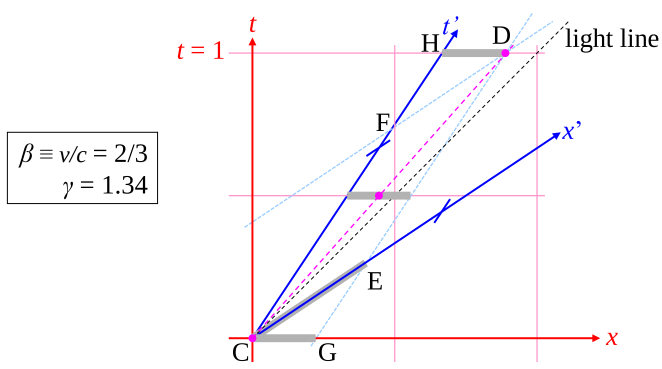


Figure 2.21 Spacetime diagram of velocity addition. CE is the initial blue position of the rod. In the blue frame, the rod doesn't move in space, but it moves in time to FD. In the red frame, the rod moves through spacetime from CG to HD.

Perpendicular Components: For *y* or *z* components of *u*', the results are almost identical. The only differences coming from eq. (2.12) for Δx. We find Δy (or Δz), noting that the frame velocity does not contribute, and there is no length contraction perpendicular to *v*:

$$\Delta y = \Delta y'\ .$$

Therefore, *all* of the velocity composition comes from the lack of simultaneity leading to Δt, eq. (2.11). This depends on u_x', so u_x' is part of the composition law for u_y' and u_z':

$$u_y \equiv \frac{\Delta y}{\Delta t} = \frac{\Delta y'}{\gamma_v \Delta t'\left(1+u'_x v/c^2\right)} = \frac{u'_y}{\gamma_v\left(1+u'_x v/c^2\right)} . \qquad (2.14)$$

Adding Velocities Quickly With Rapidity

For no good reason that I know of, people often ask, "Is there some parametrization of velocity such that composed velocities simply add?" In one dimension, yes. To do so, we must find some kind of "addition" formula (or composition formula) that has the form of the composition formula (2.13). Being well-versed in standard functions, we recognize that (2.13) has the form of the hyperbolic tangent addition formula. This is especially clear in geometrized units (described elsewhere) where we write all velocities as fractions of the speed of light: $v \to v/c$, $u' \to u'/c$, etc. Then (2.13) becomes:

$$\frac{u}{c} = \frac{u'/c + v/c}{1+(u'/c)(v/c)} \qquad \Leftrightarrow \qquad \tanh(a+b) = \frac{\tanh(a)+\tanh(b)}{1+\tanh(a)\tanh(b)} .$$

Therefore, we identify u'/c and v/c as hyperbolic tangents of two parameters, often called $\eta_{u'}$ and η_v:

$$\frac{u'}{c} \equiv \tanh \eta_{u'}, \qquad \frac{v}{c} \equiv \tanh \eta_v .$$

The η corresponding to a velocity v is called the **rapidity**. Then:

$$\underbrace{\tanh \eta_u}_{u/c} = \underbrace{\tanh\left(\eta_{u'}+\eta_v\right)}_{(u'/c)\circ(v/c)} = \frac{\tanh \eta_{u'} + \tanh \eta_v}{1+\tanh \eta_{u'} \tanh \eta_v} \qquad \Rightarrow \qquad \eta_u = \eta_{u'} + \eta_v .$$

In other words, when we compose two velocities relativistically, their rapidities *add*. I don't know how this is useful, in practice.

Energy For the Masses

Is a box of hot gas heavier than a box of cold gas? Or more quantitatively, what is the mass of random kinetic energy? Understanding how this works develops relativistic intuition, and helps avoid common mistakes.

We mentioned briefly in the Relativistic Rocket section that kinetic energy (KE) is treated specially in the modern relativistic formulas, but in principle, it is no different than any other energies: KE has mass $m = KE/c^2$, just like all energy does. Therefore, a box of hot gas "weighs" more than a box of cold gas. It doesn't just *appear* heavier; it *is* heavier. We now demonstrate this explicitly. We then conclude with a clarification of the word "mass," and a numerical example.

This sections uses the velocity composition (addition) property. Our result relies on a simple and robust statistical property of random motion: in the rest frame of a gas, the average particle momentum is zero. The following presentation requires some attention to detail, so take it slowly.

Recall that for a mass m moving at speed v, we classify its energy into 3 major forms:

$$\begin{aligned} E_{total} &= \text{rest-energy} + \text{kinetic-energy} + \text{potential-energy} \\ &= mc^2 + (\gamma_v - 1)mc^2 + V \qquad\qquad \gamma_v \equiv \left(1-v^2/c^2\right)^{-1/2} \\ &= \gamma_v mc^2 + V \qquad\qquad \left(\gamma_v mc^2 \equiv \text{rest-energy} + \text{kinetic-energy}\right). \end{aligned}$$

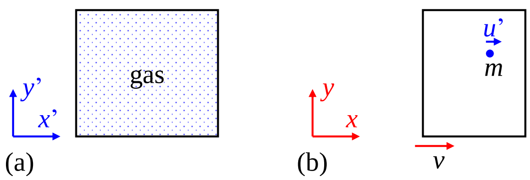


Figure 2.22 (a) A box of gas in its rest (blue) frame. (b) In the red frame, a single particle of the gas in a box.

Imagine an opaque box of gas, with a large number N of particles of mass m each (Figure 2.22a). As usual, the box is at rest in Blue's frame, and moving rightward with velocity v in Red's frame. To show that the kinetic energy of the gas (in Blue's frame) has mass $m'_{ke} = ke'/c^2$, we can show that the energy of the moving box behaves exactly like a mass of $M = Nm + m'_{ke}$. Specifically, we claim that Red should measure:

$$M = N\left[m + \left(\overline{\gamma_{\mathbf{u}'}} - 1\right)m\right] = \overline{\gamma_{\mathbf{u}'}}m \qquad \text{and} \qquad E = N\gamma_v Mc^2 = N\gamma_v \overline{\gamma_{\mathbf{u}'}}mc^2 \quad \text{(red frame)}$$
$$\textit{where} \quad \gamma_{\mathbf{u}'} \equiv \left(1 - \left|\mathbf{u}'\right|^2 / c^2\right)^{-1/2}, \text{ and } \overline{\gamma_{\mathbf{u}'}} \equiv \text{average over all } N \text{ particles}. \tag{2.15}$$

To get started, consider a single particle, with simple one-dimensional motion (x coordinate, Figure 2.22b). There is no potential energy.

To avoid a zillion factors of c in our equations, we now switch to standard geometrized units. This means we measure speeds in dimensionless fractions of c, so $v = 1.5\times10^8$ m/s $\to v = 0.5$. In geometrized units, $c = 1$ (dimensionless). $E = \gamma mc^2 \to E = \gamma m$. This greatly simplifies the algebra. To make everyone happy, we can put the c's back into the equations at the end (thus converting back to standard units).

At some instant, we take the particle's velocity in Blue's frame to be u'. The blue kinetic energy is:

$$ke' = \left(\gamma_{u'} - 1\right)m \qquad\qquad (c = 1, \text{dimensionless}).$$

For comparison, what is the red energy of the particle in the moving box?

Note that measuring the box from Red's frame does *not* involve accelerating the box, or disturbing it in any way. We're simply considering a different observer measuring the same, undisturbed box. We use relativistic velocity composition ("addition") to find u, the particle's velocity in Red's frame. Then $E_{particle} = \gamma_u m$, where m is just the (rest) mass of the particle, and $\gamma_u \ne \gamma_v$. For 1D motion, velocities compose as:[ref??]

$$u = \frac{v + u'}{1 + vu'}.$$

Both u' and v are signed velocities and can be positive or negative. Then the gamma factor for the particle, in the red frame, is:

$$\gamma_u = \left(1 - u^2\right)^{-1/2} = \left[1 - \left(\frac{v + u'}{1 + vu'}\right)^2\right]^{-1/2}.$$

We rearrange the quantity in square brackets, in hopes of comparing to (2.15):

$$1 - \frac{(v + u')^2}{(1 + vu')^2} = \frac{\left(1 + \cancel{2vu'} + v^2u'^2\right) - \left(v^2 + \cancel{2vu'} + u'^2\right)}{(1 + vu')^2} = \frac{1 - v^2 - u'^2 + v^2u'^2}{(1 + vu')^2}$$
$$= \frac{\left(1 - v^2\right)\left(1 - u'^2\right)}{(1 + vu')^2} \tag{2.16}$$

Substituting into the previous equation:

$$\gamma_u = (1+vu')\underbrace{\left(1-v^2\right)^{-1/2}}_{\gamma_v}\underbrace{\left(1-u'^2\right)^{-1/2}}_{\gamma_{u'}} = (1+vu')\gamma_v\gamma_{u'} . \tag{2.17}$$

This has some resemblance to (2.15): they both contain the term $\gamma_v\gamma_{u'}$. (Can we say something physical about this term??)

For the whole "box" of this 1-dimensional gas, we sum over all N particles:

$$E = \sum_{i=1}^{N}\gamma_{u',i}m = \sum_{i=1}^{N} m\gamma_v\gamma_{u',i}\left(1+vu'_i\right) = m\gamma_v N\overline{\gamma_{u'}} + m\gamma_v v\sum_{i=1}^{N}(\gamma_{u',i})(u'_i) .$$

$\gamma_{u',i}$ depends only on the magnitude of u'_i. And the u'_i are symmetrically distributed in $+x/-x$ (and also in $+y/-y$, and $+z/-z$). For every positive u'_i, there is a canceling negative u'_i, so the final sum on the RHS is zero. In other words, the vu' term cancels, and on restoring the factor of c^2, we recover the 1D version of (2.15).

Including y and z motion yields a simple and satisfying result. Using (2.14), the composition rule for perpendicular components y and z:

$$u^2 \equiv |\mathbf{u}|^2 = u_x^{\ 2} + u_y^{\ 2} + u_z^{\ 2} = \frac{(v+u'_x)^2 + u'^{\,2}_y/\gamma_v^{\ 2} + u'^{\,2}_z/\gamma_v^{\ 2}}{(1+vu'_x)^2}, \qquad \text{and}$$

$$\gamma_u = \left(1-u^2\right)^{-1/2} = \left[\frac{(1+vu'_x)^2 - (v+u'_x)^2 - u'^{\,2}_y/\gamma_v^{\ 2} - u'^{\,2}_z/\gamma_v^{\ 2}}{(1+vu'_x)^2}\right]^{-1/2}$$

The first two terms combine, as in (2.16):

$$\gamma_{\mathbf{u}} = (1+vu'_x)\left[\left(1-u'^{\,2}_x\right)/\gamma_v^{\ 2} - u'^{\,2}_y/\gamma_v^{\ 2} - u'^{\,2}_z/\gamma_v^{\ 2}\right]^{-1/2}$$

$$= (1+vu'_x)\gamma_v\left[1-u'^{\,2}_x - u'^{\,2}_y - u'^{\,2}_z\right]^{-1/2} = (1+vu'_x)\gamma_v\gamma_{\mathbf{u'}}$$

Compared to the 1-dimensional result (2.17), we simply replace u'_x with the vector **u'**. Our full, 3D energy is now:

$$E = \sum_{i=1}^{N} m\gamma_v\gamma_{\mathbf{u}',i}\left(1+vu'_{x,i}\right) = m\gamma_v N\overline{\gamma_{\mathbf{u}'}} + m\gamma_v v\sum_{i=1}^{N}\gamma_{\mathbf{u}',i}u'_{x,i} .$$

Again by symmetry, the last term cancels, because $\gamma_{\mathbf{u}',i}$ depends only on $|u'_x|$, and in the (blue, primed) rest-frame of the gas, the velocities have inversion symmetry along the x-axis.

The mass of kinetic energy in the box is KE/c^2, just like all other energy.
A box of hot gas is heavier than a box of cold gas.

This result is independent of the mass of the box.

Conclusions

This result has application to nuclear physics, because a significant fraction of the measured mass of a nucleon (proton or neutron) is believed to be due to the kinetic energy of its constituent quarks.[ref??]

You may have heard that "mass" is a relativistic (Lorentz) invariant, and for a particle, it is. For an opaque box, though, we can only define "mass" as that which is *measured*. For example, any physical box contains (negative) binding energy in its molecules, and when we weigh the box, that energy reduces the mass of the box. We do *not* define the mass of the box as the total rest-mass of all its constituent subatomic particles; we define its mass as that which we measure on the box as a whole. If the box is opaque, we don't know what is inside it, nor whether its measured mass is due to various forms of energy. All we know is the mass we measure, which includes the kinetic energy of any particles inside the box. *That mass*, as we showed above (did we??), is a relativistic invariant.

As a numerical example, we compute the size of this mass for a monatomic gas, say atomic hydrogen (at the temperatures of interest, hydrogen will be dissociated into separate atoms). For a single atom:

$$KE = \frac{3}{2}kT, \qquad m'_{ke} = \frac{KE}{c^2} = 2.3\times 10^{-40} \text{ kg K}^{-1}$$.?? This uses NR KE. Relativistic KE?

The temperature required to increase the mass of a hydrogen atom by 1% is about 70 billion kelvins. (At this temperature, we wouldn't really monatomic hydrogen; we'd have a fully ionized plasma of protons and electrons.)

Comments on Particle Distributions

It is possible to have an instantaneous state where the total particle momentum is zero, but the average velocity is not. However, momentum is conserved, and velocity is not. In equilibrium, the *time-average* velocity must also be zero, because zero total momentum means the box is not moving, on average. In terms of probability distribution functions, in the blue frame, each component (*x*, *y*, and *z*) of both momentum and velocity are symmetric about zero:

$$\underbrace{\text{pdf}_{u'}(u') = \text{pdf}_{u'}(-u')}_{\text{velocity distribution}} \qquad \text{and} \qquad \underbrace{\text{pdf}_{p'}(p') = \text{pdf}_{p'}(-p')}_{\text{momentum distribution}} .$$

Therefore, $\langle u'_x\rangle = \langle u'_y\rangle = \langle u'_z\rangle = 0$, and $\langle p'_x\rangle = \langle p'_y\rangle = \langle p'_z\rangle = 0$.

In the red frame, the box is length contracted, and therefore the distribution of *x*-velocities around *v* is noticeably different than the distributions of *y*- (and *z*-) velocities around 0.

Comment on Geometrized Units

Geometrized units are defined to make the speed of light $c = 1$ (dimensionless). There are two equivalent ways to do this. First, we can measure time in seconds, and distance in light-seconds (3×10^8 m). Or more commonly, we measure distance in meters, and time in units of the time it takes light to travel 1 meter; i.e., we measure time in meters, where 1 m of time ≡ (1 m)/*c* = 3.3 ns. Either way, in geometrized units, $c = 1$ (dimensionless).

Lorentz Transformations

Physical Illustration of the Spatial Transformations

Assume the usual setup of frames S (red) and S' (blue), with Blue moving toward +*x* relative to Red, at speed *v*. Figure 2.23 shows the physics of the spatial transformations between blue and red coordinates.

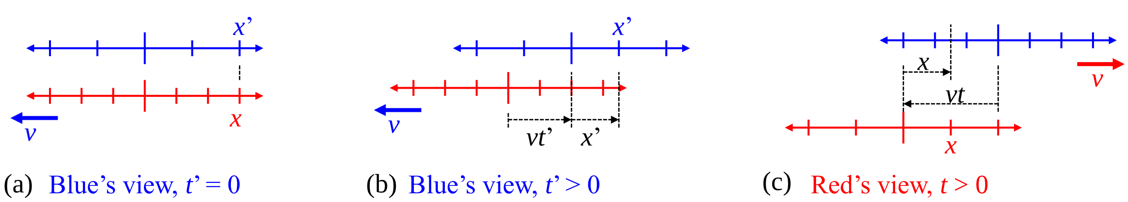


Figure 2.23: Red and blue frames, $\gamma = 1.5$. (a) In the blue frame, red rulers are contracted. $t' = 0$. (b) At arbitrary time t'. (c) In the red frame, blue rulers are contracted.

In Figure 2.23a, Blue computes how Red will measure *x*-coordinates. Blue finds the red rulers shortened, so it will take more red rulers to cover a distance *x*', by a factor γ. Thus at $t' = 0$, Red's coordinate of a blue point x' is $\gamma x'$.

In Figure 2.23b, $t' > 0$, so the frames have moved, and Red's coordinate for x' increases. Blue computes the distance from Red's origin to the point x', and then accounts for Red's shortened rulers:

$$x = (x' + vt')\gamma ,$$

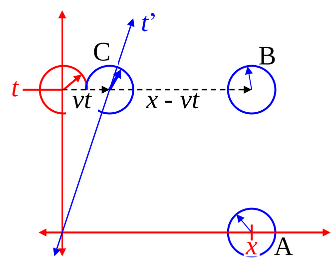
our first Lorentz transformation equation.

Figure 2.23c shows Red's view. She measures blue rulers as short, but the same basic arguments apply. We have followed convention and chosen v as a magnitude, so we must put in a minus sign to account for the difference in direction of motion of the "moving" frame. Red measures the distance from Blue's origin to the point x, and then accounts for Blue's shortened rulers:

$$x' = (x - vt)\gamma .$$

Note that these two equations are essentially identical, differing only in the plus/minus signs put in "by hand." With v a proper signed velocity of the moving (primed, blue) frame, the equations would both have negative signs. These identical transformation equations reflect the principle of relativity: with no preferred inertial frames, there can only be *one* set of transformation equations that apply to all frames. That is the equation immediately above.

Physical Illustration of the Time Transformation



$\beta \equiv v/c = 1/3$
$\gamma = 3/\sqrt{8} = 1.06$
$\theta = \tan^{-1}\beta = 18.4°$

Figure 2.24: Time transformations for $t = 0$ (clock A), and arbitrary t (clock B).

We already derived part of the time transformation in deriving (2.10) for the skew between two moving clocks at different locations. At the common origin of the red and blue frames, $t = 0$, $x = 0$ and $t' = 0$. Now consider $t = 0$, but an arbitrary location x (Figure 2.24, clock A). Use (2.10) with $2L \to x$, and remember that clocks receding from us (here, clocks to the right) have earlier times:

$$t'_A = diff' = -\gamma_v xv / c^2 .$$

For arbitrary (t, x) (clock B), we first find t' on clock C, then skew it over to clock B. Clock C runs slower than red time (t) by the factor γ_v, $t'_C = t/\gamma_v$. The distance to clock B is $(x - vt)$, so again using (2.10):

$$t'_B = t/\gamma_v - \gamma_v (x - vt)(v/c^2).$$

Simplifying:

$$t'_B = \gamma_v \left(\frac{t}{\gamma_v^2} - \frac{(x - vt)v}{c^2} \right) = \gamma_v \left[t\left(1 - \frac{v^2}{c^2}\right) + \frac{v^2 t}{c^2} - \frac{xv}{c^2} \right] = \gamma_v \left(t - \frac{xv}{c^2} \right).$$

Lorentz Transformation Matrices

Section TBS: Properties of the Lorentz transformation matrix: We start with simple Lorentz transformations. A **proper** transformation does not mirror space. An **orthochronous** transformation keeps time moving forward. Lorentz transformations include **boosts**, i.e. one observer moving in a straight line with respect to the original observer, and *rotations*, which are ordinary 3D rotations of the entire system. A boost has 3 parameters: x, y, z components of velocity. A rotation also has 3 parameters, such as the Euler angles which describe the rotation of a rigid body.

Every Lorentz transformation can be written as the composition of a pure boost and a pure rotation. Therefore, 6 real parameters define a Lorentz transformation.

Let S be an inertial frame, and S' be an inertial frame moving to the right, as seen from S: $v > 0$. The 1D (t, x) Lorentz transformation is:

$$\left.\begin{aligned} x' &= \gamma\left(x - vt\right) \\ t' &= \gamma\left(t - vx/c^2\right) \end{aligned}\right\} \qquad \left\{\begin{aligned} x &= \gamma\left(x' + vt'\right) \\ t &= \gamma\left(t' - vx'/c^2\right) \end{aligned}\right.$$

More advanced ideas involve weirder Lorentz transformations: an **improper** transformation transforms space like a mirror: everything is reflected. In other words, an improper transformation includes a parity transformation. A **non-orthochronous** transformation reverses time. We do not consider such transformations in this work.

Why the Lorentz matrix is not unitary.

Tensors

You can only do serious relativity with tensors. See *Funky Mathematical Physics Concepts* for a complete description of tensors. We give an extremely brief overview here. The concept of tensors seems clumsy at first, but it's very fundamental, and essential.

We need Lorentz transformations simply so we can choose the most convenient reference frame. In quantum field theory, for example, we often use the center of mass (COM) frame, because the calculations are simplest there. But to compare to actual experimental results, which may be recorded in a "lab" frame (not the COM frame), we then use a Lorentz transformation from the COM to the lab frame. In general, though:

We use tensors because any result that combines tensors in a covariant way is a covariant result that is valid in any frame of reference.

Tensors allow us to tell when a result is truly universal, and when it is frame dependent. Often it is much easier to calculate a scalar in some special frame, but since it is a scalar (a rank-0 tensor), everyone in any frame will calculate the same result.

Once you get used to it, tensors are essentially simple things (though it took me 3 years to understand how "simple" they are). The rules for transformations are pretty direct: transforming a rank-n tensor requires using the transformation matrix n times. A scalar is rank-0, and doesn't use the transformation matrix at all (uses it 0 times).

A vector (four-vector, in relativity) is rank-1, and transforms by a simple matrix multiply, or in tensor terms, by a summation over indices:

$$\mathbf{a}' = \mathbf{\Lambda a} \qquad \leftrightarrow \qquad a^{\mu'} = \Lambda^{\mu'}{}_{\nu}\, a^{\nu} \qquad \leftrightarrow \qquad \begin{bmatrix} a^{0'} \\ a^{1'} \\ a^{2'} \\ a^{3'} \end{bmatrix} = \begin{pmatrix} \Lambda^{0'}{}_0 & \Lambda^{0'}{}_1 & \Lambda^{0'}{}_2 & \Lambda^{0'}{}_3 \\ \Lambda^{1'}{}_0 & \Lambda^{1'}{}_1 & \Lambda^{1'}{}_2 & \Lambda^{1'}{}_3 \\ \Lambda^{2'}{}_0 & \Lambda^{2'}{}_1 & \Lambda^{2'}{}_2 & \Lambda^{2'}{}_3 \\ \Lambda^{3'}{}_0 & \Lambda^{3'}{}_1 & \Lambda^{3'}{}_2 & \Lambda^{3'}{}_3 \end{pmatrix} \begin{bmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{bmatrix}$$

where $\Lambda \equiv$ Lorentz transformation matrix

Rank-2 example: The electromagnetic field tensor **F** is rank-2, and transforms using the transformation matrix twice, by two summations over indices, transforming both indices. In other words, you have to transform the columns and also transform the rows (the order of transformations doesn't matter). This is clumsy to write in matrix terms, because you have to use the transpose of the transformation matrix to transform the rows; this transposition has no physical significance. In the rank-2 (or higher) case, the tensor notation is both simpler, and more physically meaningful:

$$\mathbf{F'} = \mathbf{\Lambda F \Lambda}^T \qquad \leftrightarrow \qquad F^{\mu'\nu'} = \Lambda^{\mu'}{}_{\mu}\, \Lambda^{\nu'}{}_{\nu}\, F^{\mu\nu} \qquad \leftrightarrow$$

$$\begin{bmatrix} F^{0'0'} & F^{0'1'} & F^{0'2'} & F^{0'3'} \\ F^{1'0'} & F^{1'1'} & F^{1'2'} & F^{1'3'} \\ F^{2'0'} & F^{2'1'} & F^{2'2'} & F^{2'3'} \\ F^{3'0'} & F^{3'1'} & F^{3'2'} & F^{3'3'} \end{bmatrix} = \begin{pmatrix} \Lambda^{0'}{}_0 & \Lambda^{0'}{}_1 & \Lambda^{0'}{}_2 & \Lambda^{0'}{}_3 \\ \Lambda^{1'}{}_0 & \Lambda^{1'}{}_1 & \Lambda^{1'}{}_2 & \Lambda^{1'}{}_3 \\ \Lambda^{2'}{}_0 & \Lambda^{2'}{}_1 & \Lambda^{2'}{}_2 & \Lambda^{2'}{}_3 \\ \Lambda^{3'}{}_0 & \Lambda^{3'}{}_1 & \Lambda^{3'}{}_2 & \Lambda^{3'}{}_3 \end{pmatrix} \begin{bmatrix} F^{00} & F^{01} & F^{02} & F^{03} \\ F^{10} & F^{11} & F^{12} & F^{13} \\ F^{20} & F^{21} & F^{22} & F^{23} \\ F^{30} & F^{31} & F^{32} & F^{33} \end{bmatrix} \begin{pmatrix} \Lambda^{0'}{}_0 & \Lambda^{1'}{}_0 & \Lambda^{2'}{}_0 & \Lambda^{3'}{}_0 \\ \Lambda^{0'}{}_1 & \Lambda^{1'}{}_1 & \Lambda^{2'}{}_1 & \Lambda^{3'}{}_1 \\ \Lambda^{0'}{}_2 & \Lambda^{1'}{}_2 & \Lambda^{2'}{}_2 & \Lambda^{3'}{}_2 \\ \Lambda^{0'}{}_3 & \Lambda^{1'}{}_3 & \Lambda^{2'}{}_3 & \Lambda^{3'}{}_3 \end{pmatrix}$$

In general, you have to transform every index of a tensor, each index requiring one use of the transformation matrix. Notice also that:

In tensor notation, the summation indices are explicit (vs. implicit in matrix notation).

This means that in tensor notation, we can write the factors in any order (vs. matrix notation where order matters):

$$\Lambda^{\mu}{}_{\nu}\, a^{\nu} = a^{\nu} \Lambda^{\mu}{}_{\nu}, \qquad \Lambda^{\alpha}{}_{\mu}\, \Lambda^{\beta}{}_{\nu}\, F^{\mu\nu} = \Lambda^{\alpha}{}_{\mu}\, F^{\mu\nu} \Lambda^{\beta}{}_{\nu} = F^{\mu\nu} \Lambda^{\alpha}{}_{\mu}\, \Lambda^{\beta}{}_{\nu}\,.$$

Four Vectors

Since relativity mixes time and space into a single manifold on which physics takes place, the vectors in relativity are four-vectors. **Four-vectors** have 3 "spatial" and one "time" component. For example, displacement is written as displacement from the origin in both time and space:

$$\mathcal{O} \equiv \begin{pmatrix} t=0 \\ x=0 \\ y=0 \\ z=0 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = (0,0,0,0)^T, \qquad x^{\mu} \equiv \left(x^0, x^1, x^2, x^3\right)^T = \left(ct, x, y, z\right)^T.$$

(We drop the transpose notation from now on.) Note that the time-component of the 4-vector has a factor of c in it. In this simple case, this makes all the components have the same units. (In other coordinate systems, they won't: polar coordinates use radius and angle, with units of meters and radians.) However, the factor of c is necessary construct physical invariants. The factor of c is somewhat inconsistent, though. For displacement, as above, we *multiply* the time component by c. This is also true of, say, electric charge-current density:

$$j^{\mu} = \left(c\rho, \mathbf{j}\right) = \left(c\rho, j_x, j_y, j_z\right).$$

Again, in cartesian coordinates, all four components have the same units. In contrast, the energy-momentum 4-vector has its time component (energy) *divided* by c, which again gives all four components the same units (in cartesian coordinates):

$$p^{\mu} = \left(E/c, \mathbf{p}\right).$$

As always when using vectors, our "reference frame" affects the coordinates we use, so the four components of a four-vector are "frame-dependent," aka "observer dependent."

3 Special Relativity Shorts

The Three Parts of Energy

Landau and Lifshitz define the rest-mass of a moving body as part of its kinetic energy [L&L p??]. In contrast, some references define rest-mass as part of its *potential* energy, since it is not, of itself, energy of motion (kinetic). There is benefit to considering all three terms of a particle energy separately: kinetic energy, rest energy, and potential energy. First, for a free particle (i.e., no potential energy):

$$(\text{kinetic} + \text{rest})^2 \equiv E^2 = c^2\mathbf{p}^2 + c^4 m^2 .$$

Then we include potential energy:

$$\text{kinetic} + \text{rest} + \text{potential} \equiv E \qquad \Rightarrow \qquad (E - V)^2 = c^2\mathbf{p}^2 + c^4 m^2 .$$

Note that **p** is the *kinetic* momentum, $\mathbf{p} = \gamma m\mathbf{v}$, *not* canonical momentum. In QFT of electrodynamics (i.e., QED), '**p**' stands for canonical momentum, so we must rewrite the above:

$$(E - V)^2 = c^2\left(\mathbf{p}_{can} - e\mathbf{A}\right)^2 + c^4 m^2, \qquad e \equiv \text{particle charge, for } e^-, e < 0 .$$

Recall that lagrangians (even relativistically) always include a term "$-V$." In Field Theory, the mass term of the lagrangian has a negative coefficient, making it "look like" a potential energy. For example, the Klein-Gordon lagrangian density is:

$$\mathcal{L}(t,\mathbf{x}) = \frac{1}{2}\left(\partial_\mu \phi(t,\mathbf{x})\right)^2 - \underbrace{\frac{1}{2}m^2\phi^2(\mathbf{x})}_{\text{mass term}} \quad \textit{where} \quad \eta \equiv diag(1,-1,-1,-1) .$$

However, this term is m^2, not m, so it is *not* a potential energy.

Nuclear Energy: Isn't That Special?

Does nuclear energy really convert mass into energy? Yes, but in exactly the same way that burning coal converts mass into energy, or discharging a battery converts mass into energy. In each case, the potential energy of the source is converted into useful work. The potential energy of the spent fuel (including its residue) is then less than the potential energy of the original fuel. Since all energy has mass, the spent fuel + residue weighs less than the original fuel, the difference being what was converted into energy.

All conversion of potential energy into useful work involves decreasing the energy source's mass.
Nuclear energy is *not* special.

Perhaps the only special feature of nuclear energy is that both fusion and fission involve creating and annihilating particles. Neutrons convert to protons, or vice-versa, and electrons and neutrinos are created. In this sense, we might say that nuclear reactions involve converting fundamental forms of matter (particle types) as well as converting forms of energy. But there is no direct conversion of "matter into energy."

In contrast, chemical batteries or burning fossil fuel breaks chemical bonds, but does not create or destroy any nucleons or fundamental particles.

The Twin Effect in Brief

SR is a system of *dynamics* that applies to inertial *observers*. The things being observed can be inertial or not; it doesn't matter.

Many people think SR only applies to things moving at constant speed,
but it would not be a theory of dynamics if nothing could accelerate.

Homer and Ulysses are twins. Homer stays home; Ulysses travels far and fast, turns around, and returns home. At the end, Homer is old, and Ulysses is young. Homer is an inertial observer, so we can compute the effect in Homer's frame with SR. Then simple time dilation reveals that Ulysses clock runs more slowly than Homers, or equivalently, Homer ages faster than Ulysses. Simple.

An observer can perform simple experiments to determine objectively whether she is inertial or not.

If I let a ball go in front of me, and it doesn't accelerate relative to me, then I am inertial.

All observers agree on which observers are inertial.

This is a non-quantitative "invariant". The stay-at-home twin (Homer) is inertial, and everyone agrees on that. The traveling twin (Ulysses) is not inertial, and everyone agrees on that (including himself). Therefore, Homer can use SR, and easily calculate the age difference.

Strictly speaking, an observer must release 3 non-collinear balls near herself to see if she is inertial. If none of them accelerate relative to her, she is inertial. This covers the unlikely case where an observer is revolving around an axis, and releases a ball or two exactly on the axis of rotation, where it would remain stationary..

Being non-inertial (i.e., accelerated), Ulysses cannot use SR, but can use GR. GR is more complicated and beyond the scope of SR. However, the final result is, of course, the same. For now, we do not explain the observations of the accelerated twin (Ulysses). See the GR section below for more information.

Relativistic Lagrangian Mechanics

Whole section TBS.

Free particle: $L = -mc^2/\gamma$. Seems complicated just to make a particle move in a straight line. An infinite number of other Lagrangians would give straight-line motion, including $mv^2/2$, and even $L = 0$??. However, a Lagrangian also has to yield the proper total energy (in systems that meet the total-energy conditions. See *Funky Classical Mechanics Concepts*). Also, the free-particle lagrangian anticipates more advanced physics, to which we now turn:

Particle in a potential: $L = -\dfrac{mc^2}{\gamma} - V(\mathbf{r})$. As always, the coefficient of V is –1, so the hamiltonian yields the total energy (again, in systems that meet the total-energy conditions).

Charged particle in EM field: $L = -\dfrac{mc^2}{\gamma} - V(\mathbf{r}) + e\mathbf{v}\bullet\mathbf{A}(\mathbf{r}) \qquad \textit{where} \quad e \equiv \text{charge}$.

The Metric System: Connecting the Dots

Relativity involves taking dot-products (aka inner products) of relativistic vectors. Unfortunately:

There are 3 different conventions for writing and computing with relativistic vectors.

We describe here these 3 conventions, and some of their implications, summarizing with an example written in nonrelativistic and all 3 relativistic notations. We finish with a quick look at vector derivatives.

Relativistic dot-products are similar to ordinary 3D vector dot-products. All dot-products produce scalars from two vectors, and are linear in both vectors. [Dot-products are always defined to be symmetric, so that **a**·**b** = **b**·**a**.] Relativistic vectors have 4 components, and are called **4-vectors**. A 4-vector in this form is usually written with superscripts for components:

$$\left(a^0, a^1, a^2, a^3\right) \equiv a^\mu, \quad \mu = 0,1,2,3$$

Dot-products in relativity are slightly different than ordinary 3-vector dot-products:

$$(3D) \qquad \mathbf{v}\bullet\mathbf{w} = \left(v_1\hat{\mathbf{x}} + v_2\hat{\mathbf{y}} + v_3\hat{\mathbf{z}}\right)\cdot\left(w_1\hat{\mathbf{x}} + w_2\hat{\mathbf{y}} + w_3\hat{\mathbf{z}}\right) = v_1 w_1 + v_2 w_2 + v_3 w_3 = \sum_{i=1}^{3} v_i w_i$$

$$(4D) \qquad a^\mu \bullet b^\mu = \left(a^0\hat{\mathbf{t}} + a^1\hat{\mathbf{x}} + a^2\hat{\mathbf{y}} + a^3\hat{\mathbf{z}}\right)\cdot\left(b^0\hat{\mathbf{t}} + b^1\hat{\mathbf{x}} + b^2\hat{\mathbf{y}} + b^3\hat{\mathbf{z}}\right) \equiv -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3$$

The key difference is that the product of the time-components is *subtracted*. The above 4-vector dot-product is used by General Relativists. 3-vector indices are usually Latin letters near i; 4-vector indices are usually Greek letters near μ (though [L&L] do the opposite).

However, (like all good physical conventions) some people do relativistic dot-products differently. Particle physicists usually define the 4D dot-product with the opposite sign:

$$(4D\text{, particle}) \qquad a^\mu \bullet b^\mu \equiv +a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 \qquad \mu = 0,1,2,3$$

Note that the 4-vectors are the same as in GR;
it is the dot-product that is defined differently.

Either way, we sometimes write 4D dot-products in matrix form:

$$\left(a^0, a^1, a^2, a^3\right)\underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{relativity convention}}\begin{pmatrix} b^0 \\ b^1 \\ b^2 \\ b^3 \end{pmatrix} \qquad OR \qquad \left(a^0, a^1, a^2, a^3\right)\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{\text{particle convention}}\begin{pmatrix} b^0 \\ b^1 \\ b^2 \\ b^3 \end{pmatrix}$$

The dot-product of a vector with itself is called the squared-**magnitude** of the vector:

$$\mathbf{a}\bullet\mathbf{a} \equiv \left|\mathbf{a}\right|^2$$

The matrix in the middle of a dot-product is called the **metric**, because it provides the "measure" of a vector. Note that the "magnitude" of a 4-vector can be positive, zero, or negative.

The 3rd metric convention is largely equivalent to the GR convention, but uses subscripts, and an imaginary time-component to achieve the same end result:

$$a_\mu \equiv ia^0\hat{\mathbf{t}} + a^1\hat{\mathbf{x}} + a^2\hat{\mathbf{y}} + a^3\hat{\mathbf{z}} \equiv a_1\hat{\mathbf{x}} + a_2\hat{\mathbf{y}} + a_3\hat{\mathbf{z}} + a_4\hat{\mathbf{t}} \qquad where \quad a_4 \equiv ia^0, \text{ and } \mu = 1,2,3,4.$$

In this form, the components are written with subscripts, and run 1.. 4, rather than 0.. 3. Then we can write the 4-vector dot-product analogously to the 3-vector dot-product, without using any explicit metric matrix:

$$a_\mu \bullet b_\mu = \sum_{\mu=1}^{4} a_\mu b_\mu = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3 \qquad \text{since} \quad a_4 b_4 = \left(ia^0\right)\left(ib^0\right) = -a^0 b^0.$$

This imaginary form is sometimes called the "east coast metric." The (+1, –1, –1, –1) metric matrix is sometimes called the "west coast metric."

Pros and cons: Here are some pros and cons of each convention. We elaborate below:

Convention	**Pros**	**Cons**
GR, diag(–1, 1, 1, 1)	4-vectors with a zero time component have the same dot-product as their 3-vector parts. This makes 4D equations look similar to their 3D equivalents.	Requires keeping track of contravariant and covariant vectors. Particle mass is given by

		$-p^\mu \cdot p^\mu$.
East coast metric (imaginary time component)	4-vectors with a zero time component have the same dot-product as their 3-vector parts (same as GR convention). No need to distinguish contravariant and covariant vectors for simple (non-derivative) vectors.	Requires keeping track of contravariant and covariant vectors (same as GR). Doesn't work for rank-2 or higher tensors. Doesn't work for derivative operators, e.g. gradient. Doesn't work for non-diagonal metrics (oblique coordinates or curved space). Fails in the case of conjugating a complex polarization vector, since the *i* from the "metric" should *not* be conjugated.
Particle physics convention (West coast metric)	Particle mass, and so-called "invariant mass," is given by $+p^\mu \cdot p^\mu$.	Reverses the sign of all 3-vector formulas.

In my opinion, the only advantage of the particle (west coast) metric is weak. However, it is used by most authors, including me, when writing about Quantum Field Theory. Also, the imaginary-time metric advantage holds only in very limited cases, so it's not worth much. This makes the GR convention, I think, demonstrably the best.

Every 4-vector has a 3-vector as its *x,y,z* components (its "spatial" components), and so is sometimes written:

$$a^\mu \equiv \left(a^0, \mathbf{a}\right) \qquad \textit{where} \quad \mathbf{a} \text{ is a 3-vector: } \mathbf{a} \equiv \left(a^1, a^2, a^3\right).$$

In this form, we can write the GR and particle conventions as:

$$\text{(GR)} \quad a^\mu \bullet b^\mu = \left(a^0, \mathbf{a}\right) \bullet \left(b^0, \mathbf{b}\right) = -\mathbf{a} \bullet \mathbf{b} + a^0 b^0 \qquad \text{(particle)} \quad a^\mu \bullet b^\mu = a^0 b^0 - \mathbf{a} \bullet \mathbf{b}.$$

To simplify the notation, we often imply the metric matrix by defining so-called "covariant" vector components to be regular vectors multiplied by the metric. (These are not *new* vectors, just a new way of projecting the components of the original vector into components. See *Funky Mathematical Physics Concepts*.) Covariant vectors use a subscript (instead of superscript) for an index:

$$\text{(GR)} \qquad a_\mu \equiv \left(a_0, a_1, a_2, a_3\right) \equiv \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix} = \left(-a^0, a^1, a^2, a^3\right)$$

$$\text{(particle)} \qquad a_\mu \equiv \left(a_0, a_1, a_2, a_3\right) \equiv \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix} = \left(a^0, -a^1, -a^2, -a^3\right)$$

With these "covariant" vectors, we can use the implied summation convention: the same index, when appearing once up and once down, is summed over:

$$\text{(GR or particle)} \quad a_\mu b^\mu \equiv \sum_{\mu=0}^{3} a_\mu b^\mu = a^\mu \bullet b^\mu = a^\mu b_\mu .$$

Note that with the east-coast (imaginary time) metric, there is no need to distinguish covariant vectors from contravariant vectors. With 3-vectors, and also with the east-coast (imaginary time) metric, we use a similar implied summation convention, but all indices are subscripts, i.e. repeated indices are summed:

$$\text{(3-vector)} \quad a_i b_i \equiv \mathbf{a}\bullet\mathbf{b} = \sum_{i=1}^{3} a_i b_i \qquad \text{(east coast)} \quad a_\mu b_\mu \equiv \sum_{\mu=1}^{4} a_\mu b_\mu .$$

As an example of the use of these forms, we now compare the electromagnetic Lagrangian in 4 different, but physically equivalent notations. All 4 of these are in common use:

Nonrelativistic: $L_{EM}(x, \mathbf{v}) = q\mathbf{v} \cdot \mathbf{A}(x) - q\phi(x)$

GR: $L_{EM}(x^\mu, \dot{x}^\mu) = q\dot{x}^\mu A_\mu(x)$

West coast: $L_{EM}(x^\mu, \dot{x}^\mu) = -q\dot{x}^\mu A_\mu(x)$ (note negative sign)

East coast: $L_{EM}(x_\mu, \dot{x}_\mu) = q\dot{x}_\mu A_\mu(x)$

Derivative Operators and Covariant Vectors: You may wonder if the "imaginary-time" metric works for derivative operators, such as the gradient. We now show that it does not. In GR or particle metrics, derivative operators naturally produce covariant vectors, without using any metric. As a simple example, consider the gradient operator acting on a scalar function:

$$\text{Let} \quad f = f(t, x, y, z). \quad \textit{Then} \quad df = \frac{\partial f}{\partial t} dx^0 + \frac{\partial f}{\partial x} dx^1 + \frac{\partial f}{\partial y} dx^2 + \frac{\partial f}{\partial z} dx^3 = \nabla f \cdot dx^\mu$$

$$\textit{where} \quad c = 1, x^0 = t$$

Note that all 4 terms are positive. There is no negative term from any metric. Now:

$$\text{Let} \quad g_\mu(t, x, y, z) \equiv \nabla f(t, x, y, z) = \left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right), \qquad i.e., \; g_0 = \frac{\partial f}{\partial t}, \; g_1 = \frac{\partial f}{\partial x}, \; etc.$$

$$\text{Then} \quad df = g_\mu dx^\mu$$

The last line shows that the gradient naturally produces a covariant vector, which can be "dotted into" an ordinary vector to compute *df* (the change in *f*) without using any metric.

The gradient (and other derivative operators such as covariant derivative, external derivative, and Lie derivative) naturally produce covariant vectors.

How would this look in the imaginary-time convention? If we believe that we don't need to keep track of contravariance and covariance, we would write:

$$dx_\mu = (dx, dy, dz, idt), \qquad g_\mu = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, i\frac{\partial f}{\partial t} \right), \quad \text{and then}$$

$$df \overset{?}{=} \sum_{\mu=1}^{4} g_\mu dx_\mu \overset{?}{=} \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz - \frac{\partial f}{\partial t} dt \qquad \text{(No! This is wrong!)}$$

Of course, if we took the derivative with respect to ($x_4 = it$) rather than just t, we would introduce another negative sign, and arrive at the correct answer. But this means we have to introduce an extra minus sign for the time component of derivative operators, which is equivalent to keeping track of contravariant and covariant vectors. Hence:

The claimed benefit for the imaginary-time metric holds only in very limited cases.
In particular, it does *not* hold for derivative operators.

Consequences of Metric Convention for Some Common Operators

We here consider some example of common operators, and the effect of the different metric choices on them. For example, in the particle metric, the lagrangian density for a real scalar field, absent other potentials, is (moving the factor of ½ out of the way to the left):

$$2\mathcal{L} = \partial_\mu \partial^\mu \phi - m^2\phi^2 = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right)\phi - m^2\phi^2 \qquad (\text{particle metric}) .$$

What does this look like in the GR metric? The field ϕ^μ is the same in both metrics, but the covariant components have opposite signs between the two metrics. Thus:

$$2\mathcal{L} = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right)\phi - m^2\phi^2 = -\partial_\mu \partial^\mu \phi - m^2\phi^2 \qquad (\text{GR metric}) .$$

So the lagrangian in the GR metric includes a minus sign which is absent in the particle metric. This difference is purely a sign convention; the physics is identical in either case.

Invariance of d^4x

One commonly needs to integrate a Lorentz scalar over a finite 4-dimensional volume, e.g. in evaluating the action from a lagrangian density over a spatial volume and an interval of time. It turns out that such an integral is itself a Lorentz invariant (the same for all inertial observers). This invariance derives from the fact that the infinitesimal 4D volume element, d^4x, is invariant. Figure 3.1a shows a simple, rectangular element of spacetime in its rest frame.

Figure 3.1b shows a naive (incorrect) application of length contraction and time dilation: the box is shorter by $1/\gamma$, but the rest-frame clock runs slowly (as measured by the moving frame), so the *t*' clock elapses more time for the duration, by the factor γ. The product (the volume) would then be invariant.

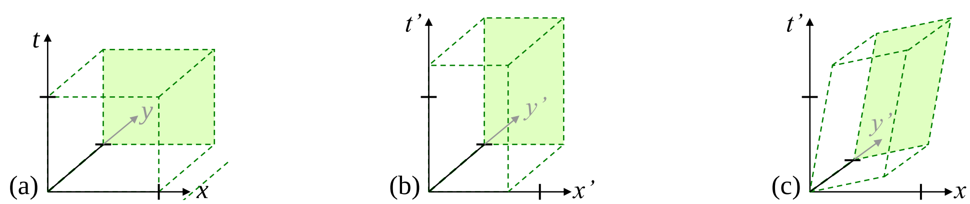


Figure 3.1 (a) A volume element in its rest frame. (b) Naive (incorrect) view of the volume element from a moving frame. (c) The correct view of the volume element.

The error in this moving element analysis is that the edges of the box are defined at a constant time *t* in the rest frame. In the moving frame, different points along the x-directed edges occur at different times *t*', so the element look like Figure 3.1c. However, this is the same volume as Figure 3.1b, because the "additional" triangle-faced volume at the top of the region exactly compensates for the "missing" triangle-faced volume at the bottom of the region, and similarly for the right and left surfaces. Thus, the volume $d^4x = d^4x'$.

4-Vectors

All 4-vectors contain a 3-vector as the spatial components, because rotations in space are included in the set of Lorentz transformations. A 3-vector is defined as a 3-component quantity that rotates like a displacement vector.

How is A^μ a 4-Vector?

TBS.

4 General Relativity Basic Concepts

We start with several important examples of relativistic physics that emphasize concepts, and require little math. These concepts clarify the applicability of SR, and also lay the foundation of GR.

One of These Twins Is Not Like the Other

The twin effect shows that an unaccelerated observer has an easier time computing things, because he can use the simple laws of SR. However, the accelerated observer can compute the same results with GR, and it is highly informative to do this calculation, as it points out several differences between GR and SR, and highlights some valid (and invalid) ways of thinking. [Thanks to Art Evans, Mark Kostuk, and Adam Orin for helpful discussion.]

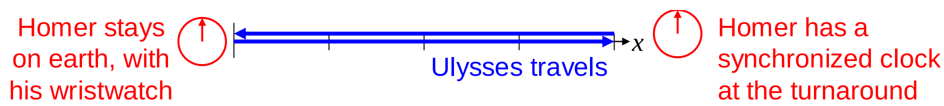


Figure 4.1 Space diagram of Homer and Ulysses.

The diagram above shows that Homer is in an inertial frame, and stays home. Ulysses leaves Homer, and travels fast for 10 Homer-years, which is only 1 Ulysses-year, i.e. $\gamma = 10$. Ulysses turns around in a short time (compared to one year), and then returns. When he meets Homer again, Homer has aged 20 years, while Ulysses has aged only 2.

We know that the simpler laws of SR apply to Homer, since he is always inertial (unaccelerated). The turn around time is negligible, so Homer computes Ulysses elapsed time using simple time dilation. Since $\gamma = 10$, twenty Homer-years = 2 Ulysses-years. QED.

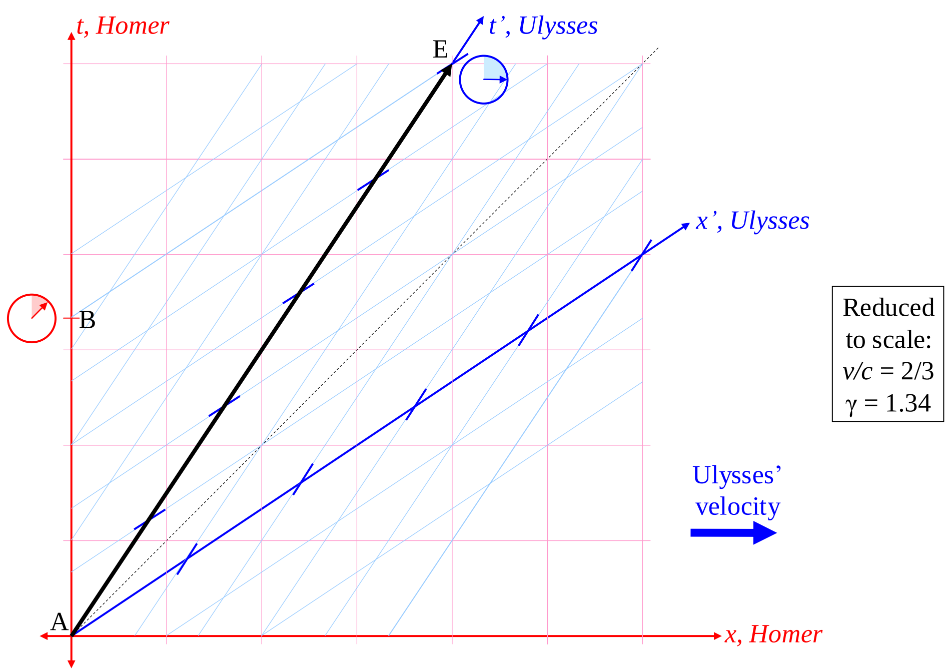


Figure 4.2 Qualitative view of Homer and Ulysses on way out (reduced scale). The red grid is Homer's frame; the blue is Ulysses. The black arrow is Ulysses' world line.

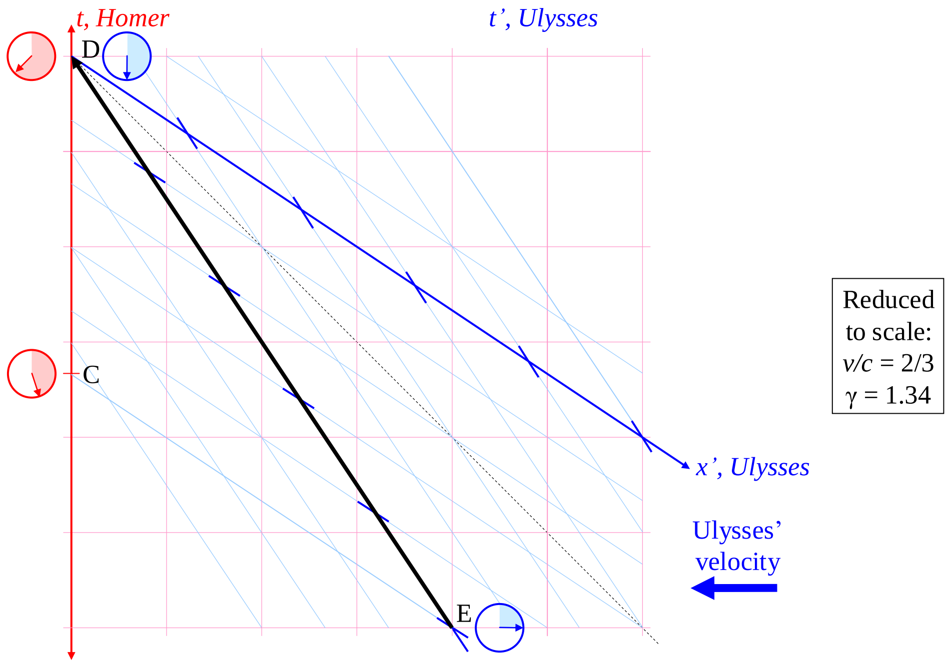


Figure 4.3 Qualitative view of Homer and Ulysses on way back (reduced scale). The red grid is Homer's frame; the blue is Ulysses. The black arrow is Ulysses' world line.

It is instructive to consider how Ulysses sees two of Homer's clocks, during the trip. The trip comprises 3 phases: (1) the way out, where both Homer and Ulysses are inertial observers; (2) the turn around, where Homer is inertial but Ulysses is accelerated; and (3) the way back, where both are again inertial. All the interesting stuff happens during the acceleration of turn around.

First: On the way out (before turn around), Ulysses is inertial, so SR applies. Recall from SR that Ulysses measures that Homer-clocks are not synchronized, and Homer-clocks approaching him have later time readings on them than Homer-clocks behind him.

When an inertial observer (Ulysses) measures another inertial set of clocks (Homer's), they read *different times*, but they all run at the *same rate*, slowed by the factor 1/γ.

On the way back (after turn around), Ulysses also measures that Homer-clocks approaching him (like the one on Homer's wrist) have later times than those behind him.

Now consider just two of Homer's clocks: the one on his wrist, and one at the turn-around point. Just before turn around, Ulysses measures the Homer-clock at the turn around point reads later than the one on Homer's wrist (because the clock is ahead of Ulysses). But after the turn around, Ulysses measures the one on Homer's wrist reads later than the Homer-clock at the turn around point. So during the turn around, the clock on Homer's wrist (as measured by Ulysses) has to gain a lot of time, compared with the clock at the turn around point.

Taking Ulysses view during the turn-around: Ulysses is accelerated, equivalent to a gravitational field pulling away from Homer. Ulysses "looks" at the clock on Homer's wrist. It is far away, and much less deep in the gravitational well. It runs very fast due to its gravitational blue-shift. The Homer-clock at the turn-around point is at the same gravitational potential as Ulysses, so during turn-around, it's gravitational red-shift is zero. It runs at essentially the same speed as Ulysses' clock, except for the small SR time dilation during turn-around (which is negligible). Thus, during turn-around, Ulysses sees the needed transition of Homer's wrist-clock changing from being earlier than the Homer-clock at turn-around, to being much later than it.

Note that even when Ulysses has zero *velocity* relative to earth, the earth clock runs much, much faster, from Ulysses' view, because Ulysses is *accelerating*, even as his velocity is zero. *Ulysses cannot use the*

SR time dilation from his accelerated frame. So long as he is accelerating, he must use GR, and compute the blue-shift because the earth clock is way up high, far shallower in the gravitational/acceleration well.

Recall that when Ulysses is not accelerating, he sees Homer's clocks as not synchronized, but they all run at the same (slow) rate. However:

When an observer (Ulysses) is accelerating, he sees inertial clocks (Homer's) running at *different rates*, depending on how deep in the potential well each one is. This is why you can't use the dynamics of SR for an accelerating observer.

A Better (Non-) Paradox

Helliwell [Hel old p??] says specifically that if Ulysses, during his turn-around, could look through a telescope to Homer on earth, Ulysses might see Homer's hair turn white in a few seconds. This is not true, and Helliwell has corrected this in his latest SR book. In relativity, we must always be careful to distinguish what we literally *see* with our eyes, from what we *measure* with our distributed system of clocks. What Ulysses *sees* is dominated by the finite speed of light, and his interception of light signals from Homer as Ulysses travels. That is *not* our concern here, and is described in [Hel new?? p]. However, Ulysses would *measure* that Homer's clock runs very fast during Ulysses' turnaround.

When Ulysses is accelerating, it is impossible for him to have synchronized clocks at earth and at the turn around.

You might think that, by reciprocity, if Homer measured Ulysses' clock, he'd measure Ulysses moving in extremely slow motion. But this leads to another (seeming) paradox: Homer has a synchronized clock at the turn-around point. If Ulysses looks slow from earth, then he must look slow from all of Homer's clocks, including the one at the turn-around point. But there, from Ulysses view, the Homer clock keeps good time, because it is at the same gravitational potential. So it would be a contradiction if Homer's synchronized clocks show Ulysses is slow (near Homer), and normal speed (near Ulysses), at the same time? The resolution is that:

Homer's and Ulysses' measurements are *not reciprocal*!

Instead, Homer measures that Ulysses' clock runs slowly the whole way out, and continues to run slowly the whole way back. Hence, Homer and Ulysses both find that when Ulysses gets home, he is younger than Homer. In fact, during the turn-around, Ulysses slows down relative to Homer, and Homer measures Ulysses clock rate increasing to ultimately agree with Homer's at the moment of zero speed.

We're already familiar with lack of reciprocity from SR: when a Red observer measures a Blue clock running slowly, it does *not* mean the Blue observer measures the Red clock running quickly. Reciprocity fails. In this case, symmetry requires that the Blue observer measures the Red clock running slowly, as well. Each measures the other is running slowly.

We can see the lack of reciprocity in the twins graphically, by combining the two spacetime diagrams from above, to put Ulysses' entire round-trip on one chart:

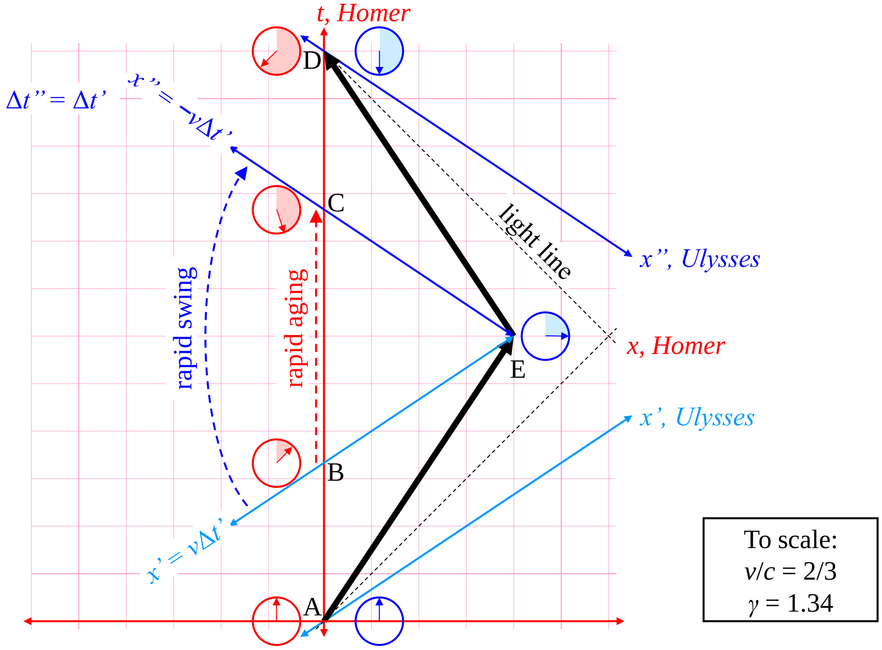


Figure 4.4 Ulysses' trip (black), with $v/c = 2/3$, $\gamma = 1.34$. The red grid is Homer's frame. The blue grid is Ulysses' frame. The black arrow is Ulysses' world line.

On the outbound leg (diagram above), Ulysses travels from A to E, along his own time axis. His position axis is *x*'. At the end of the outbound leg, just before decelerating, Ulysses reads the Homer clock at point B. As per SR, Ulysses measures that it has run slowly, compared to his (with him at point E).

Now Ulysses slows down, and (per SR) his position axis shifts toward more horizontal. At some point, Ulysses stops, and his time and position axes are orthogonal, and parallel to Homer's. Now Ulysses accelerates back toward Homer, and his position axis continues to rotate clockwise (higher on the left). At full speed, his position axis is *x*''. In summary, when Ulysses slows down and rapidly reverses course, his position axis follows the laws of SR at each instant, and rapidly swings upward on the left. In so doing, Ulysses measures Homer's clock rapidly aging (from B to C), as Ulysses' position axis rapidly swings upward. At the beginning of the return trip, Homer's clocks have now advanced more than those of Ulysses, i.e. Homer has aged more.

At constant speed on the return leg (from E to D), Ulysses again records that Homer's clock advances slower than his own. But Homer's clock started at a later time, and even though running slower, finishes still later than Ulysses' clock. Thus, after the entire round trip, Homer has aged more than Ulysses.

Again we see that analyzing the physics from Homer's inertial view allows us to use the simpler laws of SR. Taking Ulysses' accelerated reference frame requires us to use more complicated GR, but of course, the final, observer-independent result is exactly the same.

Special Relativity Implies Curved Space

Since SR deals with unaccelerated observers, we generally think of it as occurring in flat space. Some physicists even call flat (Minkowski) space the space of SR. Therefore, it is perhaps surprising that SR *implies* curved space. The standard example of a rapidly turning merry-go-round demonstrates this. An inertial observer (say, on the ground), and an observer (MGR) on the merry-go-round, watches it turn. The observers have accurate measuring rods (rulers).

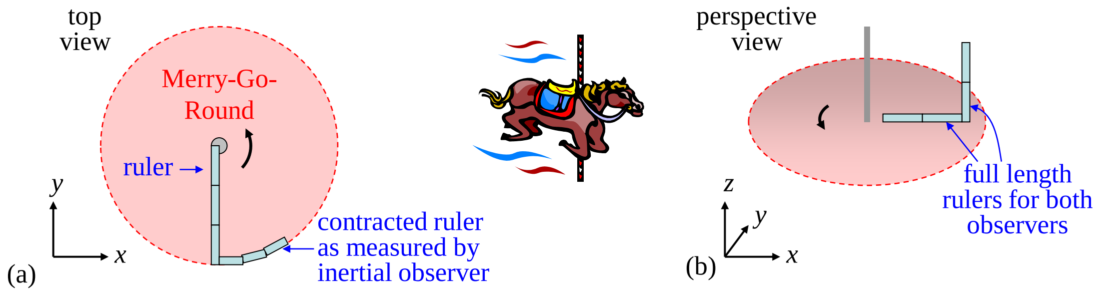


Figure 4.5 (a) The merry-go-round plus special relativity implies space in the x-y surface curved. (b) However, the x-z and y-z surfaces are flat.

Both observers measure the radius *r* (Figure 4.5a), and get the same result as if the merry-go-round were stopped, because the motion of the rods is perpendicular to their measuring length, and they are not contracted along their length. You might be concerned that MGR would measure a different path as "straight" for the radius. However, MGR can stretch a plumb wire from center to edge. In the inertial frame, the centrifugal force is along the wire, so it will follow a radius. MGR feels gravity along the wire (but not perpendicular to it), so this also defines the radius for MGR. Therefore, MGR and the inertial observer agree on a radius.

Next, each measures the circumference. The inertial observer uses rulers, and finds the circle satisfies $c = 2\pi r$. For him, the space is flat.

MGR uses rulers on the merry go round. As seen by the inertial observer, the rods now move along their length, and *SR dictates* that they are contracted. Therefore, there are more of them than $2\pi r$ would require. Both observers agree on how many rulers it takes to go around the circle, since they could all be laid down simultaneously, and counted. What is real is what is measured: for the rotating observer, the circumference is *longer* than Euclidean geometry: $c > 2\pi r$. Therefore, the rotating observer measures that the plane of rotation is a *curved space*. Note that it is *not* curved *into* the vertical direction; instead, it is *intrinsically curved*, which cannot be easily seen in the 3D manifold of space, but it *can be* measured:

Intrinsic curvature of a manifold is deviation from Euclidean geometry that can be measured entirely from within the manifold.

There is no *need* to embed the curved manifold in some higher-dimensional space. However, sometimes we perform such an embedding to help humans visualize the phenomenon.

For completeness, if the circumference $c > 2\pi r$, the space is negatively curved. If $c < 2\pi r$, the space is positively curved. We do not pursue the reasoning or details here.

Flat and Curved Simultaneously

Let us call the platform of the merry-go-round the *x-y* plane; the vertical is the *z* direction. We have shown that SR mandates that for the rotating observer, the plane of rotation is intrinsically curved [it is negatively curved because the circumference is longer than Euclid supposed]. However, what about the *x-z* plane? Again, the rulers along the *x* direction (radial) are not contracted, nor are the rulers in the *z* direction. Both observers agree on *x-z* measurements, so the *x-z* plane is flat! So, then, is the *y-z* plane. Thus only one plane of space is curved: the *x-y* plane, while the other two are flat.

Each pair of axes has its own, independent curvature.

We will examine the Riemann curvature tensor in detail later, but this fact of curvature accounts for two of its indices: each pair of axes must have its own curvature component in the tensor. We will also see that the curvature tensor must be anti-symmetric in these two indices.

Merry-Go-Rounds to General Relativity

The equivalence principle says that all acceleration is identical. Therefore, the acceleration from the merry-go-round and the acceleration from gravity would both exhibit curved space. A small leap then says that gravity *is* curved space. Some more advanced analysis suggests that not just space is curved, but *spacetime* is curved. This is where it gets weird, partly because of the negative sign in the time-time component of the metric. Much more on this later.

Reference Frames in General Relativity

Similar, to SR, a reference frame is a set of spacetime points, each labeled with coordinates, and such that a massive observer (or clock) could be at rest at any space point. It is convenient to imagine that clocks are scattered all over space, wherever they are needed for any measurement. However, in GR, identical clocks may run at different rates, and cannot be synchronized. It is usually possible to define an origin of time, $t = 0$, that all clocks agree on, but they may diverge after that due to different rates.

For example, a clock in a GPS satellite is higher up in the earth's gravitational potential, and runs faster than an identical clock on earth. *Time itself* runs faster at the satellite than on earth. We might send an isotropic pulse from earth to all the GPS satellites (all at the same altitude), and use that pulse to define $t = 0$. Later, sending another pulse, clocks on earth will be different than clocks at the satellites, because these clocks run at different rates. In practice, GPS clocks are built on earth to run slowly, by just enough so that when the speed up at altitude, they will run at the same rate as earth clocks. Thus, GPS clocks are designed to measure time in earth-clock seconds, which are longer than actual GPS seconds.

Speaking of merry-go-rounds: what about **rotating reference frames**? There are many practical situations where a rotating reference frame is desirable. For example, most of us live on the earth, and guess what: it's rotating! We can extend the rotating earth reference frame out into space. In this frame, the entire universe is rotating around us, and we are not rotating. We are accelerated, though: when we measure our acceleration with test bodies, we find gravity from earth, and centrifugal and Coriolis accelerations.

In SR, observers must be inertial, but in GR, observers can be accelerated.

For some problems, we define a "rotating" reference frame, but without any massive body like the earth. So we're out in intergalactic space, and the universe is rotating around us. In this frame, there is no central gravity from a massive body, but there are still centrifugal and Coriolis accelerations.

But rotating reference frames have limits: far from the axis of rotation, the tangential velocities of massive bodies in the universe exceed the speed of light. No observer can be at rest at those radii, and no clock can exist there to measure time. It is not meaningful to ask what an observer at such a place would measure for anything. Thus rotating frames are spatially limited to radii where the tangential speed of relevant massive bodies $< c$. (Of course, we can define a rotating *coordinate system* that extends arbitrarily far, and has coordinate points that move faster than light. But we can't *calculate physics* in any reference frame fixed to those coordinates. In principle, we *could* calculate physics in some other, valid, reference frame, and we could convert those calculations into our faster-than-light coordinate system, but that is not the same as having a faster-than-light reference frame.)

As second example, similar to rotating frames, recall the situation of a linearly accelerated reference frame: an observer is accelerated at constant proper acceleration. Such an observer has an event horizon behind her: no signal from behind the horizon can ever reach her. However, things live and exist beyond her horizon. Question: in her reference frame, can she calculate what happens in experiments beyond her horizon? No: she can't make any measurements there, and can't use GR to calculate anything there.

Tempus Fugit: Example of the g_{tt} Component of the Metric

Understanding the physical meaning of the metric is fundamental to all of relativity. We illustrate here the physical meaning of the g_{tt} component of the metric tensor field. (The other components follow by a simple extension.) We show that g_{tt} at a point $x^\mu = (t, x, y, z)$ defines how much time elapses for an observer at that point during one unit of time coordinate increase.

Consider a large spherically symmetric mass (possibly a black hole), and a far away observer (Farrah). Farrah is far enough that the force of gravity is essentially zero, and therefore the gravitational potential is approximately constant with distance. Now imagine some clocks (and observers) distributed along the radius from the observer to the mass (shown below).

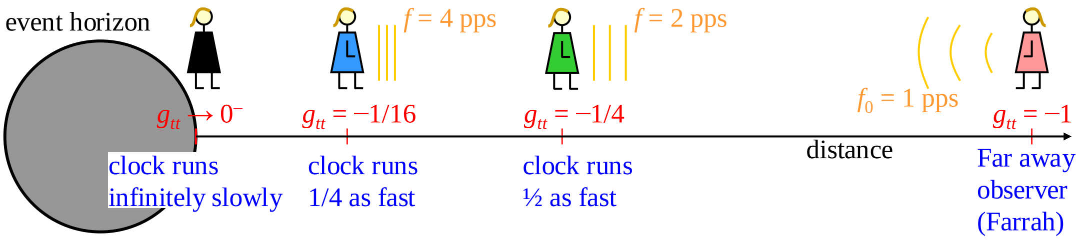


Figure 4.6 Clocks run more slowly in a gravitational potential well (as seen by a far away observer). Therefore, $|g_{tt}|$ decreases in a well.

As with all relativistic claims, we must define *who measures what*. Since Farrah has no forces on her, her reference frame is particularly simple. Therefore, we choose her measure of time as our time coordinate. We can extend this to a universal time coordinate (for all observers) by having Farrah broadcast time signals at regular intervals, say one pulse per second, on her clock. The distributed observers receive these, and use them as the time coordinate. (We could have the distributed clocks compensate for the propagation delay of her signals, but since we are concerned here only with the *rate* of time advance, there is no need.)

Similarly, any other point in space can transmit regular time signals to Farrah, which she will also measure as regular. However, the *rate* of time advance she measures may be different than the observer at the signal source. We have chosen Farrah as our observer, so her reference frame and coordinates define the spacetime metric throughout all space and time. Since no one is moving, the spacetime coordinates are stationary (constant in time).

The g_{tt} component of the metric tensor field is *defined* as the number which computes the *proper* time τ, that an observer fixed at (x, y, z) will measure (with a standard clock) during one unit of coordinate time. The *defining* equation for this is

$$d\tau^2 = -g_{tt}dt^2 \qquad where \quad \tau \equiv \text{proper time}, \quad t \equiv \text{time coordinate}$$

where, by the relativistic convention, g_{tt} is chosen to be negative. (Particle physicists often reverse that convention and choose g_{tt} positive.) The above equation is a special case of the more general equation for the spacetime interval ds^2, when the clock at (x, y, z) is fixed in coordinates, so dx, dy, and dz are all zero:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu. \qquad \text{But,} \qquad dx = dy = dz = 0 \;\Rightarrow\; \qquad ds^2 = g_{tt}dt^2 = -d\tau^2 \qquad \Rightarrow$$

$$d\tau = \sqrt{|g_{tt}|}\,dt$$

The rate of a standard clock is proportional to $\sqrt{|g_{tt}|}$. When the clock is fixed in coordinates, the (magnitude of the) spacetime interval $|ds|$ is the proper time.

Back to our example system: we define our time coordinate to be the proper time measured by Farrah (a far away observer). Therefore, *by definition*, $g_{tt} = -1$ for her (because in our metric sign convention, $g_{tt} < 0$). When Farrah sends out light pulses once per second, how do other observers (closer to the mass) measure those time intervals (diagram above)?

It is well verified that clocks in a gravitational potential well run slowly, as viewed from higher potential. The Global Positioning System confirms this every minute of every day.

Therefore, observers closer to the mass would measure a smaller (proper) time between light pulses. The pulses occur faster.

$\sqrt{|g_{tt}|}$ is the factor that converts coordinate time into proper time: it is the *gravitational* time dilation factor.

If the local clock runs ½ as fast as Farrah's clock, then $g_{tt} = -1/4$. If the local clock runs 1/4 as fast as Farrah's, $g_{tt} = -1/16$.

As an aside, if the mass is a black hole, then near the event horizon, clocks run infinitely slowly (according to Farrah), which means $|g_{tt}| \to 0$. Since $g_{tt} < 0$, we have $g_{tt} \to 0^-$, i.e. g_{tt} approaches zero from the negative direction.

[??Review stop here]

5 Building Up General Relativity

Topology, Geometry, and the Universe

Torus, and No Bull: Geometries of a Donut

The question arises: Is the 2D surface of a donut intrinsically curved? This question illustrates several important concepts in differential geometry and topology. Many serious physicists have proposed unusual geometries and topologies for the universe. Understanding donuts is a prerequisite for understanding the universe, or at least, for understanding speculations about how the universe might be. The issues can be confusing, so we also give a second example: the projective plane compared to a sphere [thanks to Chad Kishimoto]. We proceed as follows:

- Brief description of a torus, and some of its topological properties.
- Topology is not geometry.
- Embedding manifolds in higher-dimensional manifolds.
- Intrinsic curvature as a product of extrinsic curvatures.
- A flat manifold can "look" curved, and a curved manifold can "look" flat.

This section may be easier if you understand intrinsic vs. extrinsic curvature. Briefly, **intrinsic curvature** of a manifold is measured entirely within the manifold, by comparing the circumference of a circle with its radius (or other, equivalent measures). If we embed the manifold in a higher dimensional manifold, then e**xtrinsic curvature** is the radius of curvature along a line through a point on the manifold, as measured in the embedding manifold. More later.

The short answer to the question is: yes, a standard donut surface is intrinsically curved, but we must be careful what we mean by "donut." Besides a standard donut, we can mathematically construct (but not physically construct) a donut-like manifold from a cylinder, and identifying the end circles. [In mathematics, **identifying** two mathematical entities means treating them as though they are the same thing; i.e., making them "identical."] Such a "glued-together" cylinder has the same topology as a donut, but a different geometry.

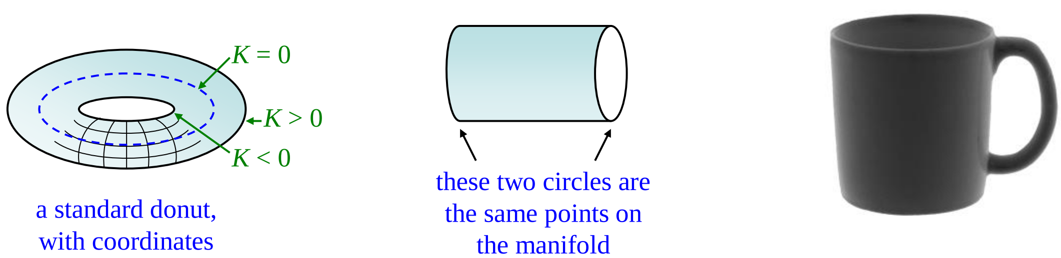


Figure 5.1 Two donut-like surfaces (tori) have the same topology, but the geometry may be different.
A coffee mug has the same topology as a donut.

Topology tells us how things are connected on a manifold, but not about distances between points on the manifold. **Geometry** (literally, "earth measure") tells us about distances between points on a manifold.

A donut topology is mathematically called a **torus** (above left). If we draw orthogonal coordinates on a donut, we see that fixing either coordinate, and traversing the other, traces out a circle. 'S' is the mathematical abbreviation for a topological circle (it's a 1-sphere), so a torus is S×S, i.e. the tensor product of two circles. This just means a torus is a manifold of pairs of numbers, each number taken as the coordinate of a point on a circle. However, a sphere is also a tensor product of two circles, so that alone does not fully describe the topology, much less the geometry. A torus has one "hole," in the intuitive sense. A sphere has no hole [and is therefore called **simply connected**]. Note that a typical coffee mug has the same topology as a donut (above, right): a 2D surface, closed, bounded, with one hole (in the handle).

The cylinder with identified end-circles (above middle) has the same topology as a donut: it is S×S, and it has one "hole." You can tell it has a hole, because you can draw a circle in the manifold around the axis, and the circle cannot be shrunk on the manifold. In contrast, for a sphere, any circle drawn on it can be shrunk to a point.

Since the coordinates we draw on a manifold are arbitrary (with some obvious restrictions that we won't belabor), *geometry* on a manifold is specified by defining a **metric tensor field**. This is a tensor at each point on the manifold, which tells us the infinitesimal distance between nearby points a and $a + da$, given their coordinate differences, da. We don't need the exact formula here, but for completeness it is:

$$ds^2 = \sum_{i,j=1}^{n} g_{ij}(a)\,(da)^i\,(da)^j \qquad \textit{where} \quad ds \equiv \text{distance between } a \text{ and } a + da$$

$$n \equiv \text{dimension of the manifold}$$

$$g_{ij}(a) \equiv \text{metric tensor at point } a$$

To help human visualization, we frequently embed a 2D manifold in our familiar, Euclidean 3D space. When we do, we can determine distances on the 2D manifold from ordinary Euclidean distances in 3D space. This allows us to define a metric on the 2D manifold which preserves these distances. Such a metric, and such distances, are the *intrinsic* geometry of the manifold: they can be determined from measurements entirely in the 2D manifold. In a sense, the 2D embedded manifold "inherits" its metric from the 3D Euclidean metric, and from the equation defining the 2D manifold in the 3D space. [Such an inherited metric is called an **induced metric**.]

Gauss proved in his Theorema egregium [ref??] that the intrinsic curvature at a point equals the inverse of the product of the minimum and maximum *extrinsic* radii of curvature at that point:

$$K = \frac{1}{R_{\min} R_{\max}} \qquad \text{Gaussian curvature}$$

Dimensionally, and in reality, curvature is a measure of something per unit area.

From the Gaussian curvature, we see that for an ordinary donut in 3 space, the curvature at points along the outer surface is positive (both extrinsic centers of curvature are in the same direction, inside the torus) (see figure above, left). The curvature at points along the inner edge of the hole is negative (one extrinsic center is inside the torus, the other is outside). Along a circle at the very top of the donut, one of the radii is infinite, and the curvature $K = 0$.

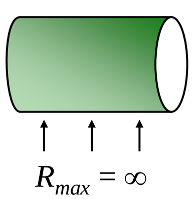


Figure 5.2 The radius of curvature along the edge of the cylinder is infinite, so the intrinsic curvature $K = 0$.

For a cylinder (above left), one direction (shown horizontal) is extrinsically straight at all points, so the intrinsic curvature $K = 0$ everywhere. This reflects that we can roll up a flat sheet of paper into a cylinder, without stretching it. [The infinite cylinder has topology R×S.] If the end circles are identified, to make a *topological* donut out of the cylinder, the horizontal edge is still straight, and therefore the curvature K is still 0 everywhere.

Hence we see that two tori, a donut and a cylinder with identified end circles, have the same topology, but different geometries. Note that we could define a coordinate mapping between the two tori, since they have the same topology. With that mapping, we could push forward the metric from one torus to the other, but the pushed-forward metric from the cylinder with identified end-circles would disagree with the "natural" metric of the donut (i.e., the induced metric from ordinary Euclidean 3-space). Similarly, pushing the metric from the donut to the cylinder would disagree with the original metric we defined for the cylinder.

Can flat be curved? We've seen that the intrinsically flat cylinder can be extrinsically curved when embedded in a Euclidean 3 space, i.e. it "looks" curved. In the next example, we see how an intrinsically curved manifold can "look" flat.

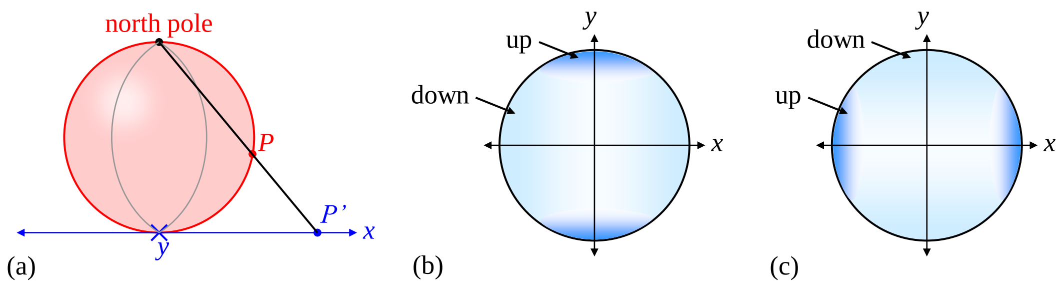


Figure 5.3 (a) The bijection between the sphere (minus the north pole) and the *x*-*y* plane. (b) A saddle pulling up on *y*, and down on *x*. (c) The saddle rotated 90°.

We can map the points on a sphere onto the entire real 2D plane. Consider a sphere, minus the north pole (a punctured sphere). The sphere sits above the real plane, with its south pole at the origin (Figure 5.3a). For every point on the sphere, draw a line from the north pole through the point until it hits the plane. This constructs a **bijection**, a 1-1 invertible mapping from every point on one manifold (the sphere) which covers every point on the 2nd manifold (the plane). The existence of such a bijection implies the two manifolds have the same topology [ref??] (I'm tempted to call them "isotopes"). This bijection also defines a coordinate transformation between any set of coordinates on the sphere, and any set of coordinates on the plane.

Now we know from our past experience that the sphere is intrinsically curved. We can see that from the extrinsic curvature of two orthogonal lines, which are always equal. Hence,

$$R_{\min} = R_{\max} \qquad \Rightarrow \qquad K_{sphere} = \frac{1}{R^2} \qquad everywhere\,.$$

The plane has the same topology as the (punctured) sphere, but it only has the same geometry if we project the metric tensor field through the coordinate transformation defined by the bijection. If we do, the metric on this plane is *not* the standard embedding Euclidean metric, and this "plane" is still intrinsically curved. Therefore, this plane and the sphere have the same topology *and* geometry; essentially, they are the same manifold. By abandoning the Euclidean metric on this plane, we have drawn a curved manifold in such a way that it "looks" flat. [I have a saying: the metric *is* the manifold, at least locally.]

Conclusions

- Specifying that a manifold is a tensor product of simpler manifolds does not fully define its topology. E.g., a sphere and a donut are both S×S, but have different topologies.
- An intrinsically flat manifold embedded in a higher dimensional space may "look" curved in its embedding, as shown by a cylinder.
- Two manifolds of the same topology can have very different geometries, as shown by a simple donut compared to a cylinder with identified end-circles.

- A bijection between two manifolds implies they have the same topology, but not necessarily the same geometry, as shown by both the donut and cylinder with identified end-circles, or by the sphere and a Euclidean plane.
- By using a non-Euclidean metric in the embedding space, we can make an intrinsically curved manifold "look" flat, as shown by pushing forward the metric from the sphere to the plane.
- Some donuts are curved; some are flat.

Aside: There are often many embedded surfaces which produce the *same* intrinsic geometry. Such surfaces are called **isometries**. For example, a saddle has negative curvature everywhere. We can embed it with the "horns" pulling up on the y axis, and "sides" pushing down on x (Figure 5.3b). If we rotate this 90 degrees (Figure 5.3c), we have a different embedding, but the curvature, and all geometry everywhere (say, in x and y), is unchanged.

The Topology of the Universe

Some serious physicists consider what might be the topology of the universe? For a long time, it was thought to be a 3-sphere (like a sphere in 4D space, not spacetime). It would then have finite volume. Today, the best evidence suggests that if it is a sphere, it radius of curvature is immeasurably large. We therefore say that it is "asymptotically flat," which means that it extends to infinity, and the matter density probably goes to zero outside the regions surrounding us.

Some physicists think the universe might have the topology of a 3-torus. A torus is perhaps more natural than it might seem at first. If you ever played the video game "Asteroids", your spaceship lived on the 2D plane of the screen. If you flew off the right side of the screen, you came back in on the left side. Similarly for top and bottom. This is the topology of a 2D torus (a 2-torus).

The universe might be a 3-torus: imagine a cube. If you move out the top, you come back in the bottom. If you move out the right, you come back in on the left. If you move out the back, you come back in on the front. In such a case, it is arbitrary where you draw the boundaries to call "front", "back", "right", etc., since the whole space is rotationally symmetric about each of the 3 axes. Amusingly, cosmologists have been simulating the universe this way for decades, not because they thought it was real, but for purely practical reasons, which involve the numerical problems of having "edges" in your simulation. The 3-torus has no edges or boundaries.

General Relativity: A Metric Theory of Gravity

A metric theory of gravity defines a metric tensor field throughout all space. The metric tensor field describes the "shape" (curvature) of space. All physics, gravitational and otherwise, occurs in the physical spacetime described by the metric tensor field. In GR, the only dynamic field is the metric tensor field (loosely analogous to the EM field in electromagnetics). But the metric tensor field is not only the dynamical field of gravity, it also determines the spacetime for all physics (including gravity).

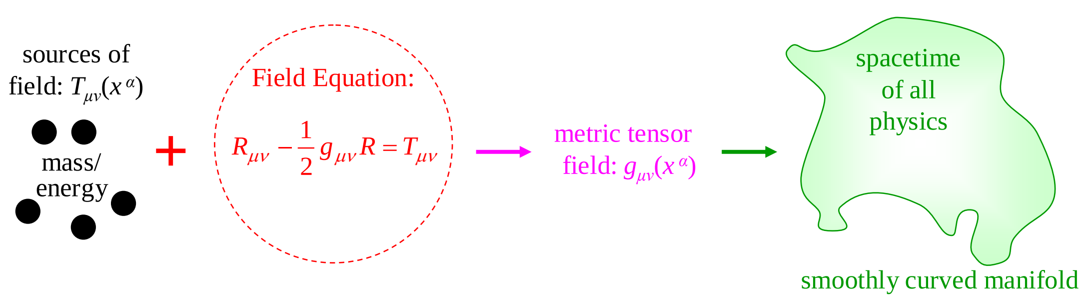


Figure 5.4 General Relativity: Just about the simplest metric theory of gravity there is.

$R_{\mu\nu}(x)$ and $R(x)$ are nonlinear functions of $g_{\mu\nu}(x)$.

The Metric Tensor Field

The **metric tensor field** quantifies intervals, which are frame independent measures of the separation between two events.

In an inertial frame (flat space), the squared-interval is the squared-distance between two events, minus the squared-distance light travels in the time between the events:

$$ds^2 = -(cdt)^2 + dx^2 + dy^2 + dz^2 = \sum_{\alpha,\beta=0}^{3} g_{\alpha\beta}dx^\alpha dx^\beta \qquad where\ (x^0, x^1, x^2, x^3) \equiv (t, x, y, z)$$

$$flat\ space: \qquad g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv \eta_{\mu\nu}$$

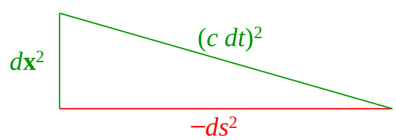


Figure 5.5 In general, the metric tensor field defines the dot product of any two vectors.

Metric Theories of Gravity

There are more general metric theories of gravity than General Relativity. By definition [Will, 1993], a **metric theory of gravity** defines a metric tensor field throughout all space. But other unobservable fields may be defined. Their only purpose is to define the metric tensor through field equations. In the end, only the metric tensor field affects observable physics.

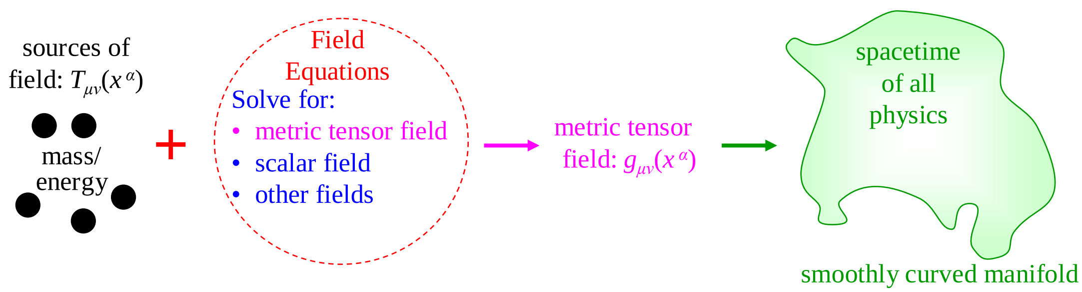


Figure 5.6 Field equations relate all the fields, to define the all-important metric tensor field.

Why GR Is Backwards

In most physics branches, problems go like this:

1. You're given a manifold (perhaps, ordinary 3-space, or the surface of a sphere)
2. You choose a coordinate system (e.g., rectangular, spherical, ...), to label the points on the manifold.
3. You work the problem in your chosen coordinates.

In contrast, GR often goes like this:

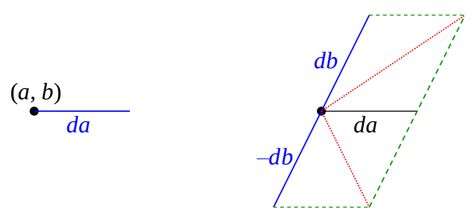
1. You're given a few properties of the manifold, such as symmetry (say, spherical for non-rotating mass, or axial, for a rotating mass).

2. You solve the relevant equations (Einstein's field equations, or a geodesic equation) for a metric $g_{\mu\nu}$, in terms of coordinates (variables) whose meaning you do not yet know.
3. From the metric, you must figure out (a) the manifold (its **geometry**, or "shape"), and (b) what the coordinates mean.

Implicit in steps 2 and 3 is that the metric *defines both* the manifold, and the coordinates. The metric is all. Of course, you can always transform one set of coordinates to another, and the metric will look different in the new coordinates, but the manifold will not. Either form of the metric (in either old coordinates, or new) will produce the *same* manifold, but the metric written in the old coordinates will label the manifold points in those old coordinates. The metric written in the new coordinates will label the manifold points in those new coordinates.

Let's see how the metric defines both the manifold, and the coordinate system on it. We'll have to use a 2-D manifold, because it's too hard to draw pictures of curved 3-D manifolds (it's even hard to draw curved 2-D manifolds). Suppose we have a metric, $g_{\mu\nu}$, in terms of coordinates (a, b). From it, we can construct the entire manifold, *and* the coordinate system on that manifold. Start with a flat, stretchy surface, on which we will draw coordinate curves, and then stretch it as demanded by the metric. We also have a ruler, so we can measure distances on the manifold. (Remember, the metric tells us the distance between any pair of nearby points.) The ruler can be *straight* and rigid, so long as it is differentially short.

Now, put a dot on somewhere on the manifold, and label it with arbitrary coordinates, say (a, b). Pick an arbitrary direction, and measure a small distance, da.



Construction of a Manifold

Now we need to find the magnitude of the coordinates b (relative to a), and the angle of the b coordinate curve. Draw a straight line segment of length 2db through the point (a, b), extending from –db to +db. At first, we let it pivot

Intrinsic geometry does not include the "shape" of the embedded manifold. Such embedding has no physical meaning. Mathematically, any two manifolds with the same metric are essentially the same manifold. The two different embeddings are called **isometries**. The orientation of a saddle (point of negative curvature) is irrelevant. Any orientation constructs the manifold; other orientations construct isometries of the same manifold.

Four Divergence

Recall that the divergence of a vector field is the net outward flow of "stuff" per unit volume, if the vector field is some kind of flow of "stuff" per unit time per unit area; i.e. if the vector field is a "flux density." In relativity, this is called a 3-divergence, for the 3 dimensions of space.

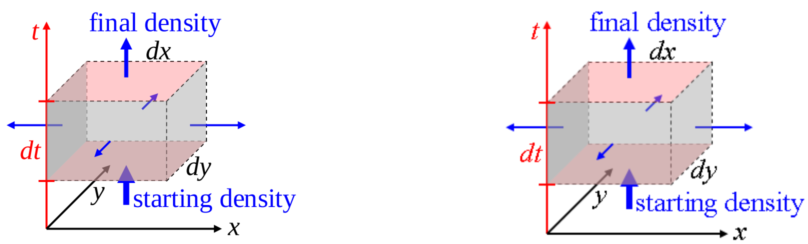


Figure 5.7 Four-divergence: The three space dimensions are shown as only two, for illustration. "Stuff" flows forward in time (upward). Some "stuff" leaks out in the spatial directions. (The two diagrams are identical, but the PDF version fails on the left side.)

A spatial density (say, charge/m^3) exists in time. On a spacetime diagram (above), the charge density flows upward through time (heavy blue arrows). During the time interval *dt*, some of the charge flows out from the spatial volume into neighboring space. Since total charge is conserved, the charge lost per unit volume in time *dt* equals the charge that flowed out in space per unit volume:

$$\rho_i - \rho_f = (\nabla \cdot \mathbf{j})\, dt \qquad or \qquad \frac{d\rho}{dt} = -\nabla \cdot \mathbf{j}$$

In this example, $d\rho/dt < 0$. This is just the continuity equation. If $d\rho/dt > -\nabla \cdot \mathbf{j}$, then charge would be being *created* in the volume over time. In general, for some density of "stuff,"

$$\frac{d\rho}{dt} + \nabla \cdot \mathbf{j} = rate\ of\ creation\ per\ unit\ volume$$

Now whenever we have a vector field that represents flow of "stuff" per unit area (i.e., flux density), we can make a 4-vector out of the "stuff" and the flux-density vector (flow of stuff per unit area):

$$(c\rho, \mathbf{j}) = (c\rho, j_x, j_y, j_z) \equiv j^\mu$$

Then we can write the rate-of-creation of stuff as

$$\frac{d\rho}{dt} + \nabla \cdot \mathbf{j} = \frac{1}{c}\frac{\partial}{\partial t} j^t + \frac{\partial}{\partial x} j^x + \frac{\partial}{\partial y} j^y + \frac{\partial}{\partial z} j^z = \frac{\partial}{\partial x^\mu} j^\mu = rate\ of\ creation\ per\ unit\ volume$$

This rate-of-creation-per-unit-volume is called the **four-divergence**. Using the Einstein summation convention, the four-divergence is usually written as

$$\frac{\partial}{\partial x^\mu} j^\mu \equiv \partial_\mu j^\mu \quad where \quad \partial_\mu \equiv \frac{\partial}{\partial x^\mu}, \ x^0 \equiv ct, \qquad \Rightarrow \qquad \frac{\partial}{\partial x^0} \equiv \partial_0 = \frac{1}{c}\frac{\partial}{\partial t}$$

And in curved spacetime, to get a true rate of change with respect to time, and true spatial derivatives (and therefore a true spatial divergence), we must use the covariant derivative:

$$\nabla_\mu j^\mu = rate\ of\ creation\ per\ unit\ volume \qquad where \quad \nabla_\mu \equiv \text{covariant derivative}$$

In general then, for conserved quantities, the 4-divergence must be zero. This is true for energy, so the rate-of-change of energy density, plus the net outflow of energy per unit volume, must be zero:

$$\frac{dw}{dt} + \nabla \cdot \mathbf{p} = \frac{\partial}{\partial x^\mu} p^\mu = 0 \quad where \quad \mathbf{p} \text{ is the flow of energy per unit time per unit area}$$

$$p^\mu \equiv (w, \mathbf{p}) = \text{energy-momentum 4-vector}$$

As usual, in curved spacetime, to get properly measured quantities, we must use the covariant derivative, e.g. conservation of energy in curved space is:

$$\nabla_\mu p^\mu = 0 \qquad \text{is conservation of energy}.$$

The Stress-Energy Tensor

The stress-energy tensor is a fundamental quantity in GR, because it is the source of all gravity. It is a set of 4 conserving currents for the 4 conserved quantities of energy, p^x, p^y, and p^z. It is a symmetric rank-2 tensor.

Essentially, all forms of energy and momentum create gravity.

The stress-energy tensor is called the energy-momentum tensor by some authors [C&W], but this could be confused with the energy-momentum 4-vector. In fact, the stress-energy tensor is the 4 conserving currents for the energy-momentum 4-vector. This section assumes you are familiar with 4-divergence and conserving currents, though we review those concepts briefly. We start with the simple example of charge conservation, then move on to energy and momentum conservation.

Every conserved quantity has a conserving current. For example, in E&M, charge is a conserved quantity. Charge is distributed in space as a charge density:

$$Q = \int_\infty \rho(\mathbf{r})\, d^3 r \ .$$

The electric current density **j** is the conserving current for charge:

$$j \equiv \frac{1}{area}\frac{dQ}{dt} \qquad \Rightarrow \qquad \frac{d\rho}{dt} = -\nabla \bullet \mathbf{j} \quad \text{or} \quad \frac{d\rho}{dt} + \nabla \bullet \mathbf{j} = 0 \ .$$

The divergence of **j** is the net outflow of charge, per unit time per unit volume. Hence, the charge density decreases by the net outflow of charge per unit time per unit volume. We can write the above **continuity equation** in relativistic form:

$$j^\mu \equiv (c\rho, \mathbf{j}) \qquad \Rightarrow \qquad \frac{1}{c}\frac{\partial j^0}{\partial t} + \frac{\partial j^1}{\partial x} + \frac{\partial j^2}{\partial y} + \frac{\partial j^3}{\partial z} = 0 = \sum_{\mu=0}^{3} \frac{\partial j^\mu}{\partial x^\mu} \equiv \partial_\mu j^\mu \equiv j^\mu{}_{,\mu}$$

With the stress-energy tensor, we are interested in conservation of energy, which is a number, like charge. We are also interested in conservation of momentum, which is a vector. But we can think of momentum, in a given reference frame, as 3 numbers: p^x, p^y, and p^z, each of which is conserved. In relativity, we combine energy and momentum into the energy-momentum 4-vector, p^μ. Then each *component* has a conserving current (if the whole vector is conserved, then each component is conserved). Therefore, we have 4 conserving currents, each of which is a 4-vector.

We now describe the stress-energy tensor in matrix form, in the hope that readers familiar with matrix operations will find it helpful. Beware that matrices have limited use to describe tensors, so this presentation may not extend to more general tensor operations (e.g., rank-3 and higher tensors). We can write the stress-energy tensor as a matrix, with the conserving 4-currents as its columns:

$$T^{\mu\nu} = \begin{bmatrix} \rho^0 & \rho^1 & \rho^2 & \rho^3 \\ \mathbf{T}^{*0} & \mathbf{T}^{*1} & \mathbf{T}^{*2} & \mathbf{T}^{*3} \end{bmatrix} = \begin{bmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{bmatrix}$$

It turns out that the stress-energy tensor can always be made symmetric, so the rows are also the conserving 4-currents.

Note that if we make a change of Lorentz frame (or *any* change of basis), we must transform each column as a 4-vector. In addition, we must transform within the columns (i.e., transform the rows) as a 4-vector, because the original energy-momentum 4-vector (for which the columns of $T^{\mu\nu}$ are conserving 4-currents) also gets transformed to the new basis. We can show this as two steps:

1. Transforming each column (i.e., among the rows of each column), taking $\mu \to \mu$'. Here, $[\Lambda]$ is the 4x4 transformation matrix:

$$T^{\mu' \nu} = [\Lambda] \begin{bmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{bmatrix} = \left[[\Lambda] \begin{pmatrix} T^{00} \\ T^{10} \\ T^{20} \\ T^{30} \end{pmatrix} \Bigg| [\Lambda] \begin{pmatrix} T^{01} \\ T^{11} \\ T^{21} \\ T^{31} \end{pmatrix} \Bigg| [\Lambda] \begin{pmatrix} T^{02} \\ T^{12} \\ T^{22} \\ T^{32} \end{pmatrix} \Bigg| [\Lambda] \begin{pmatrix} T^{03} \\ T^{13} \\ T^{23} \\ T^{33} \end{pmatrix} \right] \qquad or$$

$$T^{\mu' \nu} = \Lambda^{\mu'}{}_{\mu} T^{\mu\nu}$$

Recall that matrix multiplication from the left is distributive across the columns.

2. Transforming among the columns (i.e., transforming the rows), taking $\nu \to \nu$' (the transpose of Λ below is an artifact of matrix notation, and is not necessary in tensor notation; it has no physical significance):

$$T^{\mu' \nu'} = \begin{bmatrix} T^{0'0} & T^{0'1} & T^{0'2} & T^{0'3} \\ T^{1'0} & T^{1'1} & T^{1'2} & T^{1'3} \\ T^{2'0} & T^{2'1} & T^{2'2} & T^{2'3} \\ T^{3'0} & T^{3'1} & T^{3'2} & T^{3'3} \end{bmatrix} [\Lambda]^T = \begin{bmatrix} \left(T^{0'0} \quad T^{0'1} \quad T^{0'2} \quad T^{0'3} \right) [\Lambda]^T \\ \left(T^{1'0} \quad T^{1'1} \quad T^{1'2} \quad T^{1'3} \right) [\Lambda]^T \\ \left(T^{2'0} \quad T^{2'1} \quad T^{2'2} \quad T^{2'3} \right) [\Lambda]^T \\ \left(T^{3'0} \quad T^{3'1} \quad T^{3'2} \quad T^{3'3} \right) [\Lambda]^T \end{bmatrix} \qquad or$$

$$T^{\mu' \nu'} = \Lambda^{\nu'}{}_{\nu} T^{\mu' \nu}$$

Recall that matrix multiplication from the right is distributive across the rows.

Both steps are usually written together in tensor notation as:

$$T^{\mu' \nu'} = \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu'}{}_{\nu} T^{\mu\nu} .$$

This is a more natural notation for tensors, and the annoyance of matrix transposition does not arise. [The transposition for matrices can be deduced from the explicit summations in tensor notation.]

We now present our matrix description in a different notation. Let **T** be the 3×3 spatial submatrix of $T^{\mu\nu}$. Then:

$$p^{\mu} = \text{constant} = \left(p^0, p^1, p^2, p^3 \right) \qquad where \quad p^0 = \int_{\infty} \rho_0 \, d^3 r, \quad p^1 = \int_{\infty} \rho_1 \, d^3 r, \quad etc \qquad \Rightarrow$$

$$\left. \begin{aligned} \frac{\partial \rho^0}{\partial t} + \nabla \cdot \mathbf{T}^{*0} = 0 \\ \frac{\partial \rho^1}{\partial t} + \nabla \cdot \mathbf{T}^{*1} = 0 \\ \frac{\partial \rho^2}{\partial t} + \nabla \cdot \mathbf{T}^{*2} = 0 \\ \frac{\partial \rho^3}{\partial t} + \nabla \cdot \mathbf{T}^{*3} = 0 \end{aligned} \right\} \; where \quad \begin{aligned} &\rho^0 \equiv \textit{energy density}, \; \mathbf{T}^{*0} \equiv \text{energy conserving current density 3-vector} \\ &\rho^1 \equiv p^x \; \textit{density}, \; \mathbf{T}^{*1} \equiv p^x \text{ conserving current density 3-vector} \\ &\rho^2 \equiv p^y \; \textit{density}, \; \mathbf{T}^{*2} \equiv p^y \text{ conserving current density 3-vector} \\ &\rho^3 \equiv p^z \; \textit{density}, \; \mathbf{T}^{*3} \equiv p^z \text{ conserving current density 3-vector} \end{aligned}$$

We define the four separate conserving 4-currents in the usual way, as:

$$\left.\begin{aligned} T^{\mu 0} &\equiv \left(\rho^0, \mathbf{T}^{*0}\right) \\ T^{\mu 1} &\equiv \left(\rho^1, \mathbf{T}^{*1}\right) \\ T^{\mu 2} &\equiv \left(\rho^2, \mathbf{T}^{*2}\right) \\ T^{\mu 3} &\equiv \left(\rho^3, \mathbf{T}^{*3}\right) \end{aligned}\right\} \quad \Rightarrow \quad \left.\begin{aligned} \partial_\mu T^{\mu 0} &= 0 \\ \partial_\mu T^{\mu 1} &= 0 \\ \partial_\mu T^{\mu 2} &= 0 \\ \partial_\mu T^{\mu 3} &= 0 \end{aligned}\right\} \leftrightarrow \quad \partial_\mu T^{\mu\nu} = 0 .$$

Remember that a tensor equation with free indexes, such as ν in $\partial_\mu T^{\mu\nu} = 0$, is one equation for *every* value of the free index.

6 GR Shorts

How Far Is the Moon? The Varying Speed of Light

How far is the moon from the earth? One answer is this: lay down a bunch of meter sticks from here to the moon, along the shortest path you can make, and that gives you the distance. In principle, that is fine. In practice, it is impossible.

Here's another answer: shoot a pulse of light at a retroreflector on the moon (conveniently left by astronauts and two robot rovers), and time how long it takes to go there and come back. Then multiply by *c*. Turns out, though, this method gives a different answer than the first. Why? Because in a very real way, the speed of light is *not* constant for accelerated observers. Standing on earth, in its gravity, we are accelerated. Similarly, near the moon and its gravity, there is significant acceleration.

Consider more closely what happens to the light pulse as it travels from earth up to space, down to the moon, and back in reverse. We use our knowledge that clocks run slower deep in a gravitational potential well, and conversely, faster when shallowly in a gravitational well.

TBS. Measuring with meter sticks gives different result than timing a light pulse. Speed of light > *c* for an observer on earth.

The Principle of Relativity Implies Free-fall Is the Trajectory of Maximal Proper Time

An important result in General Relativity is that the free-fall trajectory (in spacetime) between any two events is the trajectory of maximal proper time, i.e. all "nearby" trajectories will have a shorter proper time. Some sources say that this is an *assumption* of GR, but it follows from the Principle of Relativity: physics is the same for all inertial observers. Though this sounds like a statement about Special Relativity, it also implies that free-fall trajectories in curved spacetime are those of maximal proper time. The following derivation illustrates an important method in both SR and GR: choosing a convenient frame to work in, and deriving invariant (or covariant) results.

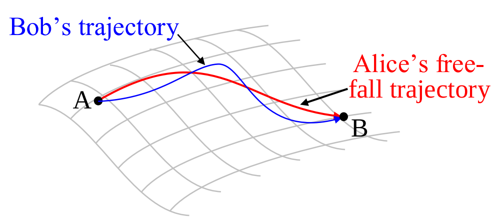


Figure 6.1 Trajectories in spacetime. Alice's is free-falling; Bob is not.

Figure 6.1 shows Alice's free-fall trajectory from event A to event B. Bob's trajectory is *not* free-fall. Who records the larger proper time? We can answer this question for *all* observers by considering how Alice would measure Bob's clock. Since Alice is in free-fall, she is inertial, and we can use Special Relativity to predict her measurements. Her inertial frame is small (local), and it follows along with her as she falls from A to B. The statement of maximal proper time is also a local statement: it applies only to nearby trajectories; therefore, we require Bob's trajectory be within Alice's local, inertial frame.

Alice sees Bob moving with respect to her, and therefore, SR dictates that she measures his clock running slowly. This is true no matter how Bob moves, changes direction, or accelerates. Therefore, at event B, Alice observes her clock to have more elapsed time than Bob's. Thus, the inertial (free-falling) trajectory has more proper time than any nearby trajectory; it is the trajectory of maximal proper time. Note that since Alice and Bob are at the same spacetime point, B, *all* observers will agree that Alice's clock shows more time elapsed than Bob's.

Embed With Schwarzschild

What is the meaning of the Schwarzschild radial coordinate, r? It is the circumference of an orbit around the source mass, divided by 2π. We can draw the curvature of space in a Schwarzschild geometry by considering a 2D slice of 3D Schwarzschild space. Consider just the equatorial plane of a Schwarzschild space ($\theta = \pi/2$). This 2D plane is spanned by Schwarzschild coordinates r and ϕ. We can reveal the plane's geometry by embedding it in an imaginary 3D space (Figure 6.2).

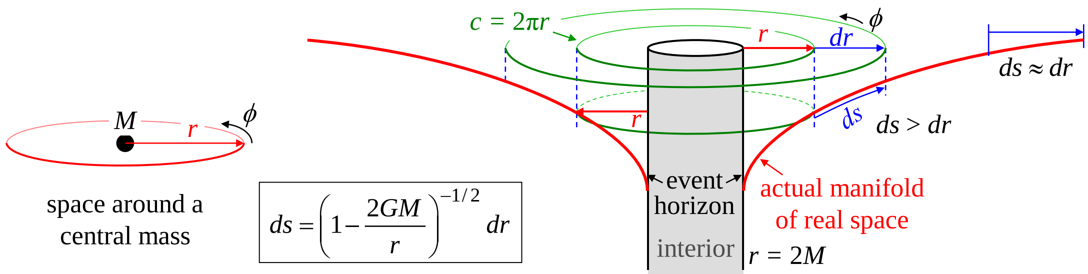


Figure 6.2 Spatial embedding surface *outside* a black hole: The circumference of a circle at Schwarzschild coordinate r is $2\pi r$. The ratio (ds/dr) becomes arbitrarily large as you approach the event horizon.

The actual manifold of real space (red) is curved. As r decreases, the real-space manifold curves "downward." If the mass is concentrated enough to be a black hold, the real-space manifold touches the "vertical" event horizon (with a vertical slope). Therefore, the horizon is a proper finite distance from any point outside the sphere. Imagine you are a particle traveling radially inward. Of course, you move along the real space manifold; the imaginary vertical axis is purely for visualization; you can never leave real space. As you approach the event horizon ($r = 2M$), in real space you are traveling straight toward the central mass, but in the embedding diagram, you are traveling "downward" on the manifold of real space. Near the event horizon, to decrease your Schwarzschild coordinate r, you must travel a longer way through space ("downward" in the embedding diagram).

By your own watch, you will reach the event horizon in a finite time.
However, to outside observers, your time becomes infinitely dilated,
and they will never see you reach the horizon.

We can never see anything get "sucked into" a black hole. For observers outside the black hole, it takes an infinite amount of time for falling matter to reach the event horizon. Instead, we just see stuff "pile up" near the event horizon.

[Sch p307b, Car p219t, F&N p157t]. This also implies that black holes cannot form in a finite amount of time, as measured by a distant observer. In other words, "If black holes *do* exist, then this is an argument that they must have been 'put in' at the beginning" [F&N p157b]. However, from our point of view, if enough matter is piled up close to the limit that would be the event horizon, then the pile-up appears (for all practical purposes) to be a black hole. We may never know the true nature of "black holes," because there is not a known way to distinguish these two cases.

Note also that:

The embedding diagram above seems to show r as measured from the "center" of the black hole.
In fact, the geometry inside the event horizon is quite different.

The embedding diagram depicts the spatial geometry *outside* the horizon. The radial coordinate r starts at the Schwarzschild radius at the surface of the event horizon, and increases from there. It is not the distance to anything. Furthermore, this embedding diagram tells us nothing about what's inside the horizon.

We can compute the function for the embedding surface by solving the differential equation for the line element. Recall from basic calculus that (in flat space) for a function $h(r)$, the line element ds is given by

$$ds = \sqrt{1 + \left(\frac{dh}{dr}\right)^2}\, dr$$

We equate this to the Schwarzschild line element, and solve for $h(r)$:

$$ds = \sqrt{1 + \left(\frac{dh}{dr}\right)^2}\, dr = \left(1 - \frac{2GM}{r}\right)^{-1/2} dr \qquad \text{(drop } ds \text{ and } dr\text{, and square)}$$

$$1 + \left(\frac{dh}{dr}\right)^2 = \left(1 - \frac{2GM}{r}\right)^{-1} = \frac{r}{r - 2GM}$$

$$\left(\frac{dh}{dr}\right)^2 = \frac{2GM}{r - 2GM}$$

$$dh = \left(\frac{2GM}{r - 2GM}\right)^{1/2} dr$$

$$h(r) = \int \left(\frac{2GM}{r - 2GM}\right)^{1/2} dr = \sqrt{2GM} \int (r - 2GM)^{-1/2}\, dr = \sqrt{8GM}\,(r - 2GM)^{1/2}$$

This is a parabola, opening to the right, and starting at the Schwarzschild radius. It clearly has finite proper distance to the horizon (the arc-length of the parabola), showing that an infalling observer reaches the horizon in a finite proper time (time measured by her).

The space manifold is called "asymptotically flat", even though it is actually parabolic, and does *not* asymptote to any flat surface. It is the *metric* which asymptotes to flat.

Black Holes

Sphericity: Note that "spherical" implies "non-rotating." A rotating star, or black hole, has an axis of rotation, which chooses a preferred direction in space. Thus "spherical" refers to the symmetry of the system, not to the shape of the object.

Inside the horizon, r is a time coordinate, and t is a space coordinate. r *decreases* with increasing time. As Carroll says, "You can no more stop moving toward the singularity [r = 0] than you can stop getting older" [Car p227t].

Remembering the Christoffel Symbol Formula

Remembering $\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right)$. TBS.

Does the Expanding Universe Accelerate Objects?

Put two objects in space, at rest with respect to each other. As the universe expands, do the objects separate? No. The expanding universe doesn't "grab hold" of matter, and can't accelerate it. The key is that for the objects to be at rest with respect to each other, they cannot both be comoving with the expanding universal coordinate system. At least one must have an initial "coordinate velocity," i.e. it must be moving wrt the coordinate system, to be stationary wrt the other body. The expanding coordinate system "slides underneath" at least one body, and the proper distance between the bodies is fixed.

However, the expanding universe *does* expand the separation of galaxies, because they are at rest with the comoving (i.e., expanding) coordinate system. The expanding universe also expands electromagnetic waves (aka photons), which is what causes the universal red-shift of radiation over time.

Can the Expansion Speed of the Universe Exceed the Speed of Light?

Yes. TBS.

Clocks, GPS, and Faster-Than-Light Neutrinos

Understanding the behavior of clocks in orbit compared to those on earth includes some surprising relativistic effects, which are often overlooked. A proper calculation illustrates several important relativistic concepts, which we now discuss. We illustrate these concepts with a realistic calculation for GPS satellites. We make the simplifying, though reasonable, approximation that the orbits are circular, and the equator is a circle.

In general, when trying to "synchronize" two clocks, there are two factors to consider: (1) whether the clocks run at the same rate (**rate synchronized**); and (2) whether one is set ahead or behind the other (**epoch synchronized**). In any calculation, we cannot determine which clock is "right;" we can only calculate how clocks will compare, and even then, we must carefully define how the comparison is performed.

A GPS satellite orbits at R_s ~ 26,600 km from the earth's *center*, at a speed of v_s = 3.9 km/s [ref??]. The earth's radius is $R_\oplus$ ~ 6370 km from its center. From these, we can compute the time dilations between earth clocks and satellite clocks. Such a calculation involves both special and general relativity for *both* the earth and satellite clocks. The satellite clocks in the GPS system are compensated for these rate-changing phenomena, so they keep accurate earth-time.

Before making a calculation, we must define our reference frames and coordinates. (Many a relativistic calculation has gone awry due to subtle changes in observers or coordinates throughout the calculation.) First, consider a ring of clocks moving in a circle around some center, e.g. a ring of GPS satellites in orbit (diagram below).

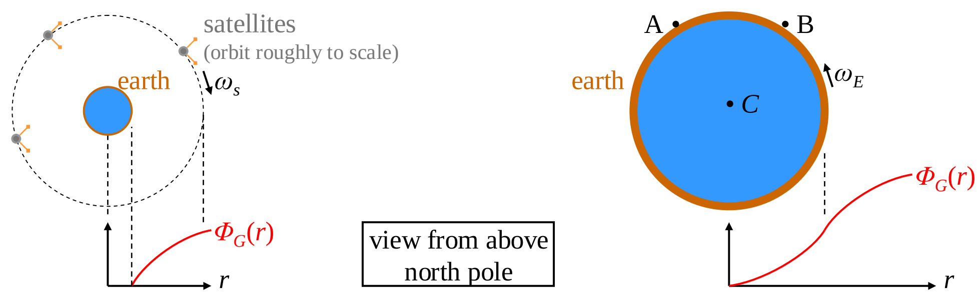


Figure 6.3 Clocks run faster than those at earth-center, according to their gravitational potential.

Cindy is a non-rotating observer, at the earth's center. [Non-rotating with respect to what? The distant stars! This is another whole topic, but for now, suffice it that the earth's equatorial bulge empirically justifies this claim.] She is inertial, and symmetrically placed, so she has a particularly simple reference frame. To her, by axial symmetry, all orbiting clocks must run at the same rate. Since they are high in orbit, GR time dilation speeds them up, but also, since they are moving relative to her stationary frame, SR time dilation slows them down. The same arguments apply to clocks on earth. Furthermore, because she can echo signals from one kind of clock to the other, earth and satellite clocks can be rate synchronized. Cindy's relaying of time signals between clocks establishes that:

For orbiting clocks, rate synchronization is transitive:
if S syncs to C, and A syncs to C, then A syncs to S.

Thus all the satellite and all earth clocks can be simultaneously rate synchronized to each other. [However, we have already seen that for linearly moving clocks, synchronization is *not* transitive: in the case of the merry-go-round, the MCRF can be synchronized to the edge observer *E*, and *E* can be synchronized to the center, but the MCRF won't be synchronized to the center.]

Still more, Cindy can epoch synchronize all the clocks by choosing a zero-point on her clock *C*, broadcasting radio time signals, and accounting for the propagation delays of EM waves to each earth and satellite clock. And finally, Cindy chooses earth-time as her time coordinate (thus her clock runs slower than the time coordinate, but no one is actually at the center of the earth, so we don't care.) This rate and epoch synchronized system of satellite and earth clocks defines our time coordinate.

Thus we have a *universal* time coordinate, whose rate and origin are defined as ordinary time on earth. This is essentially what is done by GPS receivers, LHC physicists, and everyone else. The curvature of *space* from the earth's gravity is negligible.

A crucial point is:

When the clocks are compensated for both GR and SR between the earth and satellite, there is no SR time dilation due to the relative motion of the satellites w.r.t the earth's surface. There *is*, however, a *classical* Doppler effect, from the changing distance between the clocks due to the relative motion of the satellites and the earth's surface.

All satellite clocks are always synchronized to all earth clocks *regardless* of their relative motion with respect to the earth clocks. There is *never* SR time dilation between satellite clocks and earth clocks.

Furthermore, even though one earth clock sees other earth clocks around the world as in relative motion (due to earth's rotation), they are *still* all synchronized with each other at all times, i.e. there is no SR time dilation between them.

There are, however, EM propagation delays which depend on the *position* (not speed) of the satellite clocks w.r.t. earth clocks. These lead to a *classical* (not relativistic) Doppler effect, due to the changing distance between a satellite clock and an earth clock. This Doppler effect is not important here.

We now compute the GR and SR time-dilations between earth clocks and satellite clocks, which are compensated for in the design of the satellite clocks.

Gravitational Time Dilation: In weak gravity, time dilation is well-approximated using the Newtonian gravitational potential Φ(r) [ref??]:

$$d\tau = \sqrt{|g_{tt}|}\,dt = \left(1+\frac{\Phi}{c^2}\right)dt \qquad where \quad d\tau \equiv \text{time interval measured by local observer}$$

$$dt \equiv \text{time coordinate, and}$$

$$\Phi(r) = -\int_{origin}^{r} g(r)\,dr \qquad where \quad g(r) \equiv \text{local acceleration of gravity}$$

Φ(r) is the gravitational work per unit mass required to move a mass from the origin to r. Note that a clock runs faster at high altitude because it is at a higher gravitational *potential* than the surface of the earth, *not* because the force of gravity is weaker there.

For a static point at the orbit of a satellite clock, we choose the earth's surface as the origin, and integrate simply:

$$\Phi(R_s) = -\int_{R_\oplus}^{R_s} -9.81\left(\frac{R_\oplus}{r}\right)^2 dr = -9.81R_\oplus{}^2\left[\frac{1}{R_s}-\frac{1}{R_\oplus}\right] = 4.75\times10^7 \text{ m}^2/\text{s}^2$$

$$\Phi(R_s)/c^2 = 5.28\times10^{-10} \quad or \quad 45.6\ \mu\text{s/day}$$

Though for this problem we don't need to know it, it is instructive to compute the gravitational time dilation from Cindy to the earth's surface. We assume the earth has uniform mass density, and thus the *acceleration* of gravity increases linearly with radius:

$$\Phi(R_\oplus) = -\int_0^{R_\oplus} -9.81\frac{r}{R_\oplus}\,dr = 9.81\cdot\frac{1}{2}R_\oplus = 3.12\times10^7 \text{ m}^2/\text{s}^2$$

$$\Phi(R_\oplus)/c^2 = 3.47\times10^{-10} \quad or \quad 30.0\ \mu\text{s/day}$$

Note that the earth clocks run *faster* than Cindy's, even though gravity is *stronger* at the surface, because:

Clock rates depend on the gravitational *potential*, not the gravitational acceleration.

Special Relativistic Time Dilation: Above we computed the gravitational time dilation from the earth to a static point at the orbit of a satellite. But the satellites are not static; they are moving. They follow orbits perpendicular to the force of gravity. It can be shown that bodies moving perpendicular to gravity follow SR dynamics. Therefore, when viewed from our stationary global time frame, the satellite clocks are slowed by SR time dilation.

Note that in principle, we cannot directly compute the satellite clock's SR slowing by the *relative* speed of the satellite clock w.r.t. the earth clock, since the earth clock is *not* inertial. Instead, we compute the SR slowing of both satellite and earth clocks relative to Cindy, and take the difference to find the slowdown of the satellite clock relative to the earth clock.

From Cindy's frame, the satellite clocks orbit at a speed of 3900 m/s. Then:

$$\gamma = \frac{1}{\sqrt{1-v_s{}^2/c^2}} \approx 1+\frac{1}{2}\left(\frac{v_s}{c}\right)^2 = 1+8.5\times 10^{-11} \qquad or \qquad 7.3\ \mu\text{s/day}$$

At the earth's surface, the rotational speed with respect to the fixed stars is given by one rotation per *sidereal* day (sday), which is about 3 min 56 s shorter (0.27%) than a solar day. At the equator,

$$v_e = \frac{2\pi}{86{,}164\ s/sday} R_\oplus = 465\ \text{m/s}, \qquad \gamma = 1+1.20\times 10^{-12} \quad or \qquad 0.10\ \mu\text{s/day}$$

This is small, which is good, since it varies with the earth clock's latitude. However, even 0.1 μs/day is large for GPS purposes: it's 30 m at light speed. This means that even clocks on earth must compensate for the varying effect of SR time dilation at different latitudes. (In practice, GPS receivers simply synchronize to the RF signal from the satellites, which implicitly accounts for all time dilation effects.)

Finally, the net effect of GR and SR is that the satellite clocks run 45.6 - (7.3 - 0.1) = 38.4 μs/day *faster* than earth clocks (at the equator). Again, the satellite clocks are constructed to run slowly by this much on the ground, so they run at earth-speed when in orbit.

Note, then, that if the earth clocks at the LHC and Gran Sasso are epoch synchronized, then there is no SR effect contributing to the time-of-flight (TOF) of neutrinos.

Practical Epoch Synchronization of Earth Clocks: Common View

In 2011, the Large Hadron Collider OPERA team reported that neutrinos traveling from the LHC to an observing station in Gran Sasso traveled faster than light. They measured the time-of-flight for the journey, and knowing the distance, found it was shorter than light-travel time

The neutrino propagation time deficit is ~60 ns out of ~2.4 ms, or about 25 ppm = 2.5×10^{-6} discrepancy [ref??]. Earth clocks have an accuracy on the order of 10^{-14}, so they can *easily* measure this discrepancy. However, since the launch time is measured by a clock at the LHC, and the arrival time is measured by a different clock at Gran Sasso, 730 km away, any epoch-time discrepancy between the two clocks contributes directly to a time-of-flight (TOF) error. The OPERA teams claims that the two clocks are epoch-synchronized to within 3 ns [ref??], which is well below the measured TOF deficit. To epoch synchronize the clocks, OPERA uses a National Institute of Standards and Technology (NIST) procedure called Common View Synchronization, which we now describe [ref??].

Armed with the knowledge that there exists a stationary global coordinate system which includes all satellite and earth clocks, the basic idea of Common View Synchronization is trivial. We wish to epoch-synchronize clocks at points A and B on the surface of the earth, hundreds or thousands of km apart. The two earth stations use GPS receivers to listen to a *single* (common) GPS satellite, which broadcasts its position and its (global) time. Each station receives this same signal, and notes its time of arrival. Each station knows its location, and computes the electromagnetic (EM) propagation time from the satellite to itself, and simply adjusts its clock to account for this. If both GPS receivers were identical all the way from their antenna, cabling, electronics, etc., then the two clocks would now be epoch-synchronized to within a few ns.

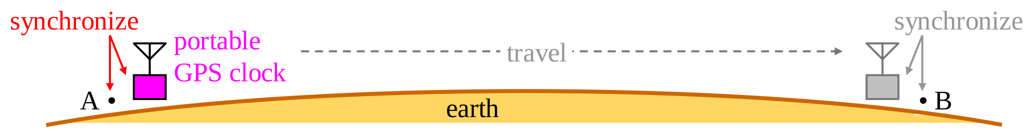


Figure 6.4 Local receiver differences are removed by synchronizing both stations to a portable receiver.

In practice, however, each station has its own antenna, cable, and receiver, all of which contain unavoidable variations from other stations, which could add up to many ns. Therefore, a practical epoch synchronization of the two earth clocks (A and B) includes an additional step. We use a third, portable, GPS receiver, which includes its own antenna, cabling, electronics, etc (see above). We then follow 3 steps:

1. We first epoch synchronize clock A to the portable clock. We record the time difference between clock A and the portable clock as an offset for clock A.
2. We carry the portable clock to clock B. The elapsed time is irrelevant, the portable clock need not keep time during this transportation, and may be powered off. We then epoch-synchronize clock B to the portable clock. We record the time difference between clock B and the portable clock as an offset for clock B.
3. Use a common view to epoch synchronize clock A and clock B, using the time-deltas for each clock as previously determined from the portable clock.

Note that the portable clock may have its own time-delta, but since it is used to calibrate both earth clocks, any fixed offset in the portable clock is incorporated equally in the two earth clocks, thus achieving epoch synchronization of the two clocks with each other, in both the inertial earth frame, and the rotating earth-surface frame. As a further check, we can return the portable clock to station A, and verify that the epoch error between the two is near zero. Again, the portable receiver need not keep time while traveling.

Note that in each of steps 1 and 2, both clocks are "nearby," so any atmospheric delays are common, and do not appear in the time-delta between the two. However, many hours or days may elapse between steps 1 and 2, and the atmosphere may change appreciably in that time. That leads to a possible time offset between clocks A and B after step 2. Step 3 then establishes epoch synchronization, because even at 730 km separation, the atmospheric effects largely cancel (the satellite orbits at 20,200 km above the earth's surface).

We conclude that there exists a stationary global time coordinate that allows all clocks to be simultaneously rate and epoch synchronized. Common View Synchronization works, as described by NIST. There are no SR time dilation effects to be accounted for during epoch synchronization of earth clocks. Thus, the cause of the neutrino TOF deficit is not attributable to SR effects in clock epochs or rate.

No Such Thing As a Rigid Body

If there were, it could be used to communicate faster than light. Why is communicating faster than light bad? Because it allows effects to precede causes.

Similarly, it's impossible to spin-up a disk without it deforming (no rigid disks). Even after settling to steady-state rotation, this resolves Ehrenfest's "paradox:" relativity itself induces stresses in rotating materials. In practice, centrifugal forces would destroy the object long before relativistic stresses are detectable [Dieks 2010 p14].

Dieks, Dennis, *Space, Time and Coordinates in a Rotating World*, arXiv:1002.0130v1.

The Covariant Derivative of the Metric Tensor is Zero

I think the vanishing of the covariant derivative of the metric tensor field is built into the definitions of the metric tensor and parallel transport. This is true for any metric manifold, whether physical spacetime or not. To see this, note that if we transport an arbitrary vector from position x to $x+dx$, then by definition (of parallel transport) its magnitude is the same. Furthermore, such magnitudes are measured by the metric tensors at x and $x+dx$. This equality of magnitude is true for all vectors (in all directions). A straightforward theorem of tensors says that if two tensors produce the same results when acting on all vectors, then the tensors are equal. Hence, the metric tensor at x equals (geometrically) the metric tensor at $x+dx$.

However, when we write the two value of the metric tensor, at x and *at* $x + dx$, in some basis, the *components* of the metric tensors may be different. This is true for all tensors: a difference in components does not imply a difference in the tensors; it may be due to a difference in the basis vectors at x and $x + dx$.

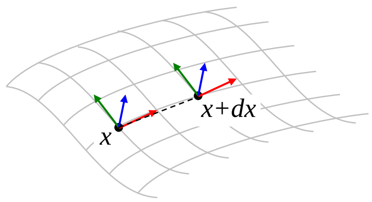


Figure 6.5 3D example showing the covariant derivative of the metric tensor field is zero. The three vectors point in the 3 directions of the manifold, and are parallel transported to $x+dx$.

7 Gravitomagnetism

Gravito-magnetism is the orphan-child of physics: hardly anyone has heard about it, but it is as fundamental as electro-magnetism. It is a purely relativistic effect, not in Newtonian gravity. But the same reason why Coulomb's law ($F = q_1q_2/r^2$) can't be true (it claims action at a distance), is also true of Newtonian gravity ($F = m_1m_2/r^2$). And just as Special Relativity implies electro-magnetism from the static limit of Coulomb's law, SR implies gravito-magnetism from the static limit of Newtonian gravity.

Gravitomagnetism is sometimes misleadingly called "frame dragging." This is a misnomer, because frame-dragging suggests that parts of space are "pulled along" by gravity, like a stream dragging leaves along its flow. But the direction of gravitomagnetism is just like electro-magnetism: $\mathbf{F} = \mathbf{v} \times \mathbf{B}_g$. The direction of acceleration of a body depends on the direction of its velocity. In a stream, you don't find leaves being pulled in different directions depending on their direction of motion.

> Gravity includes a velocity-independent force (Newton) and a velocity-dependent force (gravitomagnetic), closely analogous to the electric and magnetic fields in E&M.

There is renewed interest in gravito-magnetism with the launch of Gravity Probe B.

Lorentz invariance implies a gravitomagnetic field [1, p3]

"Any theory that combines Newtonian gravity together with Lorentz invariance in a consistent way, must include a gravitomagnetic field, which is generated by mass current." [1]

A $1/r^2$ force law is not Lorentz invariant. Since Lorentz transformation includes velocity, any $1/r^2$ force must be accompanied by a source-velocity-dependent field. Given the structure of the Lorentz transformation, the velocity dependent field must be a Biot-Savart-like magnetic field (to within a constant factor).

The term "frame-dragging" is deprecated. For a moving source-mass, the direction of the gravitomagnetic force depends on the direction of the velocity of the *test-mass*. Just like the Lorentz magnetic force, the gravitomagnetic force is perpendicular to the test-mass velocity. There is no "frame" being "dragged" in any particular direction. There is a gravitomagnetic field, much like an electro-magnetic field.

> Gravitoelectric and gravitomagnetic forces are two components of gravity, but not the only two, so even weak gravity is more complicated than electromagnetism.

Give example of another component??

Linearized Gravity

Nonlinear equations are hard to solve

Use perturbation theory:

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + h_{\mu\nu}, \qquad \text{where} \quad \left| h_{\mu\nu} \right| << 1$$

$$\text{Define} \quad \bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\left(h^{\mu}{}_{\mu}\right)$$

$\bar{h}$, instead of h, just makes the equations simpler.

The Gravitomagnetic Field

Use perturbation theory to compute the weak-field, non-relativistic perturbation to the metric:

$$\left(-\frac{\partial^2}{\partial t^2}+\nabla^2\right)\bar{h}_{\mu\nu}=16\pi G T_{\mu\nu} \quad \overset{vacuum}{\Rightarrow} \quad \left(-\frac{\partial^2}{\partial t^2}+\nabla^2\right)\bar{h}_{\mu\nu}=0$$

Compare to E&M (tensor vs. vector):

$$\left(-\frac{\partial^2}{\partial t^2}+\nabla^2\right)A^\mu=4\pi j^\mu \quad \overset{vacuum}{\Rightarrow} \quad \left(-\frac{\partial^2}{\partial t^2}+\nabla^2\right)A^\mu=0$$

Can jump right to gravity waves; but let's not.

Use the perturbed metric to compute the equations of motion. (Solve the geodesic equation.) Gravitomagnetic term:

$$\mathbf{a}_i=kM_j\mathbf{v}_i\times\frac{\left(\mathbf{v}_j\times\hat{\mathbf{r}}_{ij}\right)}{r^2} \quad \Rightarrow \quad \mathbf{B}_G(\mathbf{r})=-kM\frac{\left(\mathbf{v}\times\hat{\mathbf{r}}\right)}{r^2}$$

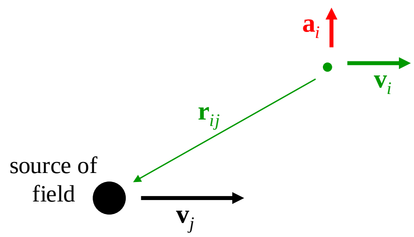


Figure 7.1 Left hand rule. Compare to Biot-Savart: $\mathbf{B}(\mathbf{r})=\frac{q}{c}\frac{\left(\mathbf{v}\times\hat{\mathbf{r}}\right)}{r^2}$

Where Did the Tensor Go?

To order ($1/c^2$), only the first row and column of h are significant:

$$\bar{h}\sim\begin{pmatrix} O\left(\frac{1}{rc^2}\right) & O\left(\frac{1}{r^2c}\right) & O\left(\frac{1}{r^2c}\right) & O\left(\frac{1}{r^2c}\right) \\ O\left(\frac{1}{r^2c}\right) & \sim 0 & \sim 0 & \sim 0 \\ O\left(\frac{1}{r^2c}\right) & \sim 0 & \sim 0 & \sim 0 \\ O\left(\frac{1}{r^2c}\right) & \sim 0 & \sim 0 & \sim 0 \end{pmatrix}$$

Reduces equations to vectors (rank-1 tensors).

Gravitational "Maxwell's Equations"

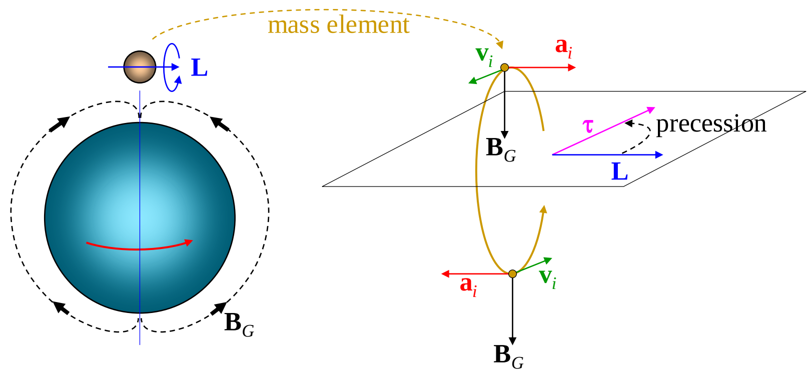
$$\mathbf{E}_G=-\nabla\Phi-\frac{1}{2c}\frac{\partial\mathbf{A}}{\partial t}, \qquad \mathbf{B}_G=\nabla\times\mathbf{A}. \qquad \textit{Lorenz gauge}: \ \frac{1}{c}\frac{\partial\Phi}{\partial t}+\frac{1}{2}\nabla\cdot\mathbf{A}=0$$

$$\nabla\cdot\mathbf{E}_G=-4\pi G\rho, \qquad \nabla\cdot\mathbf{B}_G=0$$

$$\nabla\times\mathbf{E}_G=-\frac{1}{2c}\frac{\partial\mathbf{B}_G}{\partial t}, \qquad \nabla\times\frac{1}{2}\mathbf{B}_G=\frac{1}{c}\dot{\mathbf{E}}_G-\frac{4\pi G}{c}\mathbf{j}$$

Valid for weak field, non-relativistic speeds. Imply propagating waves: gravity waves. Factors of 2 are remnants of rank-2 tensor wave equation, and spin 2 gravitons.

Example: Gravitomagnetically Precessing Gyroscopes



Use the solar system barycentric frame

Source of gravitomagnetic field is earth's spin

Precession at poles is same direction as earth spin

This is not geodetic precession; gravitomagnetism is much smaller

Gravity Probe B:

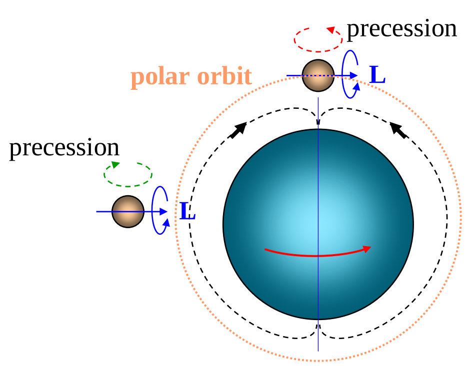
Equatorial precession opposite direction of earth spin

Partially cancels GPB signal: total precession = ¼ polar precession

Dipole approximation no good: altitude 640 km = 0.1 R□

Dipole approximation is never much good: if far enough for dipole, effect is too small to see

Do the integral: 42 mas/y is the published number



8 Gravity Waves

How Do Gravity Wave Detectors Work?

Interferometric Detectors: Interferometric gravity wave detectors use a laser interferometer to measure changes in the distance between two mirrors, brought on by the passing of a gravity wave. The diagram below illustrates how this works:

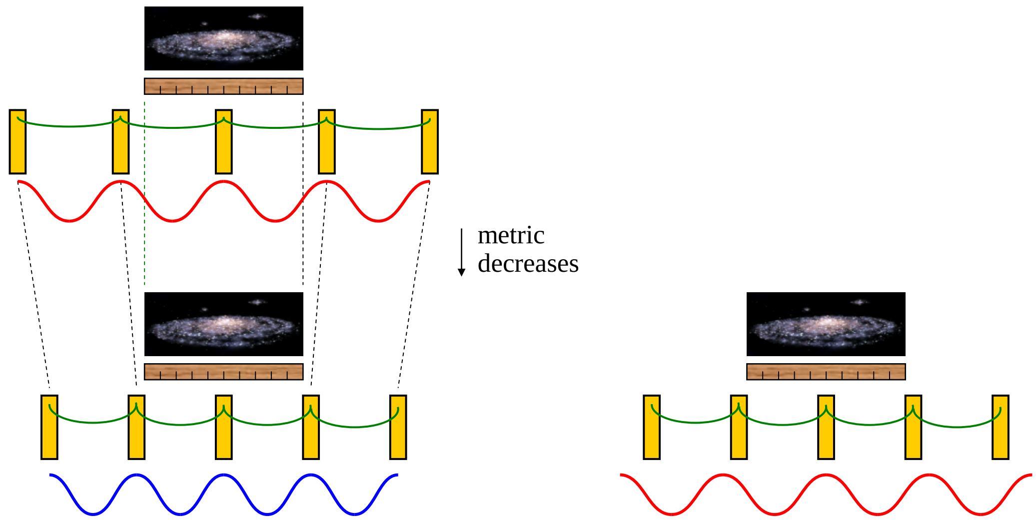


Figure 8.1 (Left) Floating masses (fence posts) mark fixed coordinate locations. If the metric shrinks, the coordinate curves (fence posts) get closer together. Any waves in-flight also get shrunk. Objects (e.g. rulers) defined by intra-body forces are not affected. (Right) The metric is now smaller, but constant in time. When new waves enter, they retain their wavelength.

The detector measures the distance between the two end posts (mirrors) with one leg of a laser interferometer. When a gravity wave passes by the detector, it changes that distance. For definiteness, we choose the shrinking part of the gravity wave (in the next half-cycle, the metric will be expanding space). After the metric has shrunk, the distance between the ends is less than before. Any laser light in-flight while the metric shrinks is also shrunk (blue shifted). However, the period of the wave is much longer than the round-trip time of the laser light. Therefore, the laser light is quickly flushed out, and replenished with new light. The now-shrunk space has no memory of its prior size, and during the time of a round-trip, the metric is nearly constant. Thus the new light is *not* shrunk. The position of the interferometer fringe now marks the new distance between the mirrors. Essentially, the method by which the mirror separation is shrunk does not matter; the interferometer always measures it.

Note that during slow metric shrinks, bodies held together by intra-body forces (i.e., bound systems) do not change size. A ruler, a rope, and a galaxy don't change size. Rulers and ropes are held together by electromagnetic forces between the atoms. Galaxies are held together by gravity. As the universe expands, galaxies don't get bigger; they get farther apart.

For higher frequency gravity waves, where the period approaches the round-trip time of the laser light, the light does not have time to be flushed and replaced with new (full-wavelength) light. Thus the returned light is alternately red- and blue-shifted, and the interference fringes cannot stabilize to the instantaneous mirror separation. This effect reduces the sensitivity of the gravity wave detector for such frequencies.

The gravity wave detector is said to detect the wave without taking any energy from the wave, because in principle, changing the separation between the mirrors does not impart any energy to them. More precisely, the energy imparted to them can, in principle, be made arbitrarily small. This is a crucial property, because it allows the detector to respond to the wave amplitude ($\sqrt{h_{\mu\nu}}$), rather than the metric perturbation itself, $h_{\mu\nu}$. This means the sensitivity drops off as 1/R from the source, instead of $1/R^2$. This slower drop off is required for the detector to have any hope of detecting known sources.

Resonant Mass Detectors: Early gravity wave detectors attempted to use large masses of aluminum (called "bars") to respond to the wave's changes in the metric. The hope was that the changing metric would stress the bar, and excite it into a small, vibration (oscillation). Clearly, such a detection method requires taking energy from the wave, and thus is expected to have a $1/R^2$ response to distance from the source. That seems to rule out resonant masses as possible detectors.

Gravity Wave Detectors and Quantized Gravity

If it is true that the interferometric detector can detect a gravity wave without taking energy from it, this may have implications for quantum gravity. In a private discussion with Daniel Holz and Kim Griest, we concluded that gravity is fundamentally different than the other 3 forces (electromagnetic, strong, and weak). Or at least, tradition implies that it is different. Some important points:

(1) The gravity wave detectors, such as LIGO, are thought to detect gravity waves, without taking any energy from the gravity wave itself. This is fundamentally different than the other 3 field theories, where interactions occur only by the exchange of particles, and therefore energy. This view of gravity is therefore inconsistent with a traditional, hypothetical "graviton."

(2) In GR, gravity is not a "force;" it is curvature of the spacetime manifold. It can be approximated as a force only in weak gravity. The other 3 forces are modeled as true forces. The simple "graviton" seems inconsistent with gravity as curvature, instead of a force.

(3) Even if we believe that gravity is quantized similarly to the other 3 forces, and yet somehow we can still detect gravity waves without energy (or particle) exchange, Quantum Field Theory says that it is the *field excitations themselves* which are quantized. The dynamical field of GR is the metric tensor field. This means that even if LIGO can detect waves without particle exchange, the waves it is detecting are still quantized. How does this affect LIGO? At this point, I think all quantized predictions are purely speculative, since we started from a contradictory premise (detecting waves without energy exchange, but nonetheless gravitons are similar to the other quantum field theories). How can one predict the effect of a quantized field that does not exchange particles, but somehow influences measurements? Nothing like this exists in physics, so I have little confidence in any speculation.

(4) Isn't the cross section of LIGO on the order of the length of the laser beam times the beam width? In other words, instead of 1 km × 1 km = 1 km², isn't the cross section more like 1 km × 1 cm = 10^5 cm²? Then we can estimated the gravity-wave power affecting LIGO from a supernova:

$$\text{Power} \equiv P = 10^{58} \text{ erg/month} \sim 3\times 10^{51} \text{ erg/s}, \qquad \text{distance} \equiv R = 10^{10}\, ly = 10^{28}\ cm \qquad \Rightarrow$$

$$I = P/A \sim 10^{-5}\ erg/s/cm^2$$

$$\sigma_{LIGO} \sim (1\,km)(1\,cm) = 10^5\ cm^2 \qquad \Rightarrow \qquad P_{LIGO} = I\sigma_{LIGO} \sim 1\,erg/s$$

If gravity is quantized in energies anything like $hf = \hbar\omega$, then for a 1 kHz wave, LIGO intercepts quadrillions of quanta per second, putting it far above the quantum limit.

Polarization Tensor

For things like gravity waves, the field which varies sinusoidally is a rank-2 tensor, which can be written as a 2D matrix (below, left):

$$\varepsilon_{\mu\nu}(t,\mathbf{x}) = \mathrm{Re}\left\{ \begin{pmatrix} \varepsilon_{tt} & \varepsilon_{tx} & \varepsilon_{ty} & \varepsilon_{tz} \\ \varepsilon_{xt} & \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yt} & \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zt} & \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} e^{-i\omega t} \right\} \qquad \varepsilon_{+}(t,\mathbf{x}) = \frac{1}{\sqrt{2}} \mathrm{Re}\left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{-i\omega t} \right\}$$

Above right is ε_{+}, the polarization tensor for "+" polarization propagating in the *z*-direction.

9 The Death of General Relativity

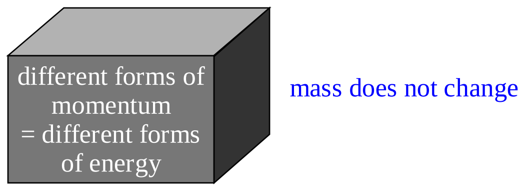
Gravito-Stern-Gerlach

TBS: Gravitomagnetism introduces a $J \cdot B_G$ energy.

Similar to Einstein's original $E = mc^2$ reasoning, all angular momentum must be the same. Otherwise, the mass of a black box could change without any outside interaction, from an internal spin-flip/orbital angular momentum interchange.

∇B_G creates a gravito-Stern-Gerlach force. This is a true force (not acceleration), independent of mass, which means different particles accelerate differently under its influence: a composition-dependent (spin-dependent) acceleration which violates the weak equivalence principle!

~10^{-28} eV. Can currently measure energies to ~10^{-17} eV, far too imprecise to detect this effect.



Neutron Interferometry

See Sakurai, *Modern Quantum Mechanics* [Sak p127].

10 Appendices

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[2] Ashby, Neil and Bahman Shahid-Saless, *Geodetic precession or dragging of intertial frames?*, Physical Review D, Volume 42, number 4, p1118.

[3] K. Nordtvedt, *Lunar Laser Ranging – A Comprehensive Probe of Post-Newtonian Gravity*, arXiv:gr-qc/0301024, 1/7/2003.

Glossary

Definitions of common GR terms:

aka "also known as"

cf "compare to." Abbreviation of Latin "confer."

comprise to include. E.g., an insect comprises 3 parts: a head, thorax, and abdomen. We could say "An insect is composed of 3 parts," but there should be no "comprised of".

coordinates a way of continuously labeling points in spacetime with real numbers.

coordinate system A set of coordinates which covers all of the space of interest.

ergo Latin for "therefore".

light cone the section from a given point of a spacetime diagram which a massive particle could reach, defined by the light-lines leading out from the point. For a representative 1-time + 2-space dimensional diagram, the light cone is a 3D cone pointing up (future) and down (past) from a spacetime point.

manifold a "space," i.e. set of points, each point of which can be labeled by a set of *continuous* real numbers (coordinates). We can define functions of points on manifolds, and differentiation of those functions with respect to the continuous coordinates of the points.

orthochronous a type of transformation that keeps time moving forward. Usually refers to Lorentz transformations.

reference frame	a state of motion in which a massive observer could exist. A "reference frame" usually includes a set of coordinates in which the observer is at rest.
space-time	deprecated form of spacetime.
spacetime	unified manifold of space and time, on which different observers would draw different coordinates, but which provides a single, universal manifold for physics.
spherical	spherically symmetric, which implies "non-rotating." A rotating sphere (e.g., a star or black hole), has an axis of rotation, which chooses a preferred direction in space. Thus "spherical" refers to the symmetry of the system, not to the shape of the object.
static	not moving, compare to "stationary." A uniformly rotating sphere is stationary, but not static.
stationary	properties constant in time. compare to "static." A uniformly rotating sphere is stationary, but not static.

Formulas

$$\left.\begin{aligned} x' &= \gamma(x - vt) \\ t' &= \gamma\left(t - vx/c^2\right) \end{aligned}\right\} \quad S, S' \to \quad \left\{\begin{aligned} x &= \gamma(x' + vt') \\ t &= \gamma\left(t' - vx'/c^2\right) \end{aligned}\right.$$

$$u^\mu = \frac{dx^\mu}{d\tau} \qquad a^\mu = \frac{du^\mu}{d\tau} \qquad f^\mu = ma^\mu = m\frac{du^\mu}{d\tau} = m\frac{d^2x^\mu}{d\tau^2}$$

Christoffel symbols: $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu} \equiv \left(\partial_\mu \mathbf{e}_\nu\right)^\rho = \frac{1}{2} g^{\rho\lambda}\left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right)$

Diagonal metric [Car 3.213-3.215, p147]:

$$\Gamma^\lambda_{\mu\nu} = 0 \ \ (\mu \ne \lambda \ne \nu), \qquad \Gamma^\lambda_{\mu\mu} = -\frac{1}{2g_{\lambda\lambda}}\partial_\lambda g_{\mu\mu} \ \ (\mu \ne \lambda), \qquad \Gamma^\lambda_{\mu\lambda} = \frac{1}{2g_{\lambda\lambda}}\partial_\mu g_{\lambda\lambda} \ \ (\forall \mu, \lambda)$$

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda = \partial_\mu V^\nu + \left(\partial_\mu \mathbf{e}_\lambda\right)^\nu V^\lambda \qquad \nabla_W V^\nu \equiv W^\mu \nabla_\mu V^\nu$$

$$\frac{D}{d\lambda} = \frac{dx^\mu}{d\lambda}\nabla_\mu, \qquad \frac{D}{d\tau} = \frac{dx^\mu}{d\tau}\nabla_\mu$$

$$\begin{aligned} R^\rho{}_{\sigma\mu\nu} &= \partial_\mu \Gamma^\rho_{\sigma\nu} - \partial_\nu \Gamma^\rho_{\sigma\mu} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\sigma\nu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\sigma\mu} \\ &= \partial_\mu \Gamma^\rho_{\sigma\nu} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\sigma\nu} - [\mu \leftrightarrow \nu] \end{aligned}$$

$$R^\rho{}_{\sigma\mu\nu} V^\sigma \equiv \left[\nabla_\mu, \nabla_\nu\right] V^\rho = \left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\right) V^\rho \qquad \text{[Car 3.112 p122]}$$

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}, \qquad R = R^\mu{}_\mu = g^{\mu\nu} R_{\mu\nu}, \qquad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

Geodesic Equation: $\dfrac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\dfrac{dx^\rho}{d\lambda}\dfrac{dx^\sigma}{d\lambda} = 0$

Geodesic Deviation: $A^\mu = \dfrac{D^2}{dt^2} S^\mu = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma$ (Car 3.208 p146)

Schwarzschild metric [Har 9.1 p186]:

$$\left.\begin{array}{l} ds^2 = -\left(1-\dfrac{2GM}{c^2 r}\right)(c\,dt)^2 + \left(1-\dfrac{2GM}{c^2 r}\right)^{-1} dr^2 \\ \qquad\qquad +r^2 d\theta^2 + (r\sin\theta)^2\, d\phi^2 \end{array}\right\} \Leftrightarrow \begin{pmatrix} -\left(1-\dfrac{2GM}{c^2 r}\right) & 0 & 0 & 0 \\ 0 & \left(1-\dfrac{2GM}{c^2 r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & (r\sin\theta)^2 \end{pmatrix}$$

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