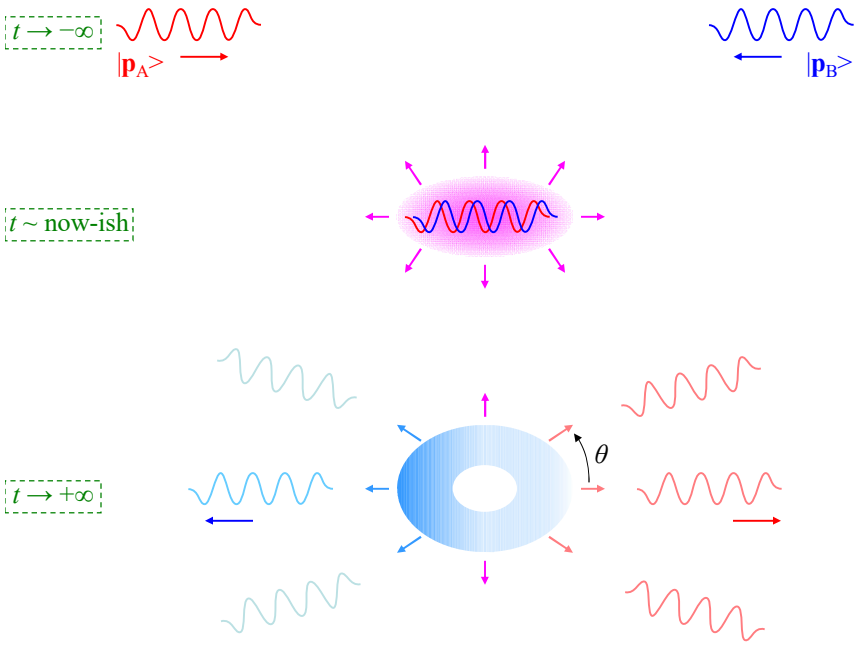
Funky Quantum Field Theory Concepts

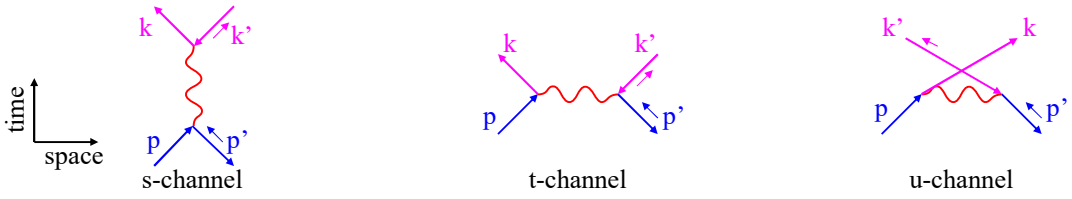
Particle physics

The Anti-Textbook*

A Work In Progress. See elmichelsen.physics.ucsd.edu for the latest versions of the Funky Series.
Please send me comments, especially for errors.

Eric L. Michelsen





"One imagines in classical physics the fictitious observer who sees everything and disturbs nothing." -- Ramamurti Shankar [p261]

* Physical, conceptual, geometric, and pictorial physics that didn't fit in your textbook.

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Physical constants: 2006 values from NIST. For more, see http://physics.nist.gov/cuu/Constants/ .

Speed of light in vacuum	$c = 299\ 792\ 458$ m s^{-1} (exact)
Gravitational constant	G = 6.674 28(67) x 10^{-11} m^3 kg^{-1} s^{-2}
Relative standard uncertainty	±1.0 x 10^{-4}
Boltzmann constant	k = 1.380 6504(24) x 10^{-23} J K^{-1} = 8.61734 x 10^{-5} eV/K
Stefan-Boltzmann constant	σ = 5.670 400(40) x 10^{-8} W m^{-2} K^{-4}
Relative standard uncertainty	±7.0 x 10^{-6}
Avogadro constant	N_A, L = 6.022 141 79(30) x 10^{23} mol^{-1}
Relative standard uncertainty	±5.0 x 10^{-8}
Molar gas constant	R = 8.314 472(15) J mol^{-1} K^{-1} = 0.0820575 L-atm/(mol-K)
calorie	4.184 J (exact)
Electron mass	m_e = 9.109 382 15(45) x 10^{-31} kg
Proton mass	m_p = 1.672 621 637(83) x 10^{-27} kg
Proton/electron mass ratio	m_p/m_e = 1836.152 672 47(80)
Atomic mass unit (amu)	1.660 538 86 × 10^{-27} kg
Elementary charge	e = 1.602 176 487(40) x 10^{-19} C
Electron g-factor	g_e = –2.002 319 304 3622(15)
Proton g-factor	g_p = 5.585 694 713(46)
Neutron g-factor	g_N = –3.826 085 45(90)
Muon mass	m_μ = 1.883 531 30(11) x 10^{-28} kg
Inverse fine structure constant	α^{-1} = 137.035 999 679(94)
Planck constant	h = 6.626 068 96(33) x 10^{-34} J s
Planck constant over 2π	$\hbar$ = 1.054 571 628(53) x 10^{-34} J s
Bohr radius	a_0 = 0.529 177 208 59(36) x 10^{-10} m
Bohr magneton	μ_B = 927.400 915(23) x 10^{-26} J T^{-1}

Other values:

1 inch ≡ 0.0254 m (exact)

1 drop ≡ .05 ml (metric system, exact. Other definitions exist.)

1 eV/particle = 96.472 kJ/mole

1 esu ≡ 1 statcoulomb = 3.335 641 × 10^{-10} C

kiloton ≡ 4.184 x 10^{12} J = 1 Teracalorie

bar ≡ 100,000 N/m^2

atm ≡ 101,325 N/m^2 = 1.013 25 bar

torr ≡ 1/760 atm ≈ 133.322 N/m^2

5 Introduction to Quantized Fields

We introduce here the quantum field, and its representations in a few bases. The entire chapter is devoted to nothing but *n*-particle wave-functions and quantum field states. We emphasize the bridge between NRQM and quantum field theory (QFT). QFT states are the foundation of all further QFT work. The quantum field theory *builds on* the quantum wave-theory; quantization does not *refute* the wave-theory. For murky reasons, the quantized field theory is sometimes called "second quantization" [Dirac 1958 p230], though we don't find that label very helpful.

We start by developing single particles, then move to multiple particles as single-excitations of multiple basis states (aka "modes"). This is sufficient for most scattering theory. We proceed to multiple excitations of boson basis states, which is needed for topics such as quantum optics, radio frequency (RF) theory, and detailed atomic structure.

This chapter seems lengthy, but it is built on just a few concepts. Most of the concepts apply to both bosons and fermions, with differences as noted along the way. Similarly, most of the concepts also apply to condensed matter systems of aggregate excitations, such as phonons in a crystal, but we do not address such systems directly.

This chapter largely stands alone: it does not rely on previous chapters, nor on any relativistic quantum physics. It requires only an undergraduate-level understanding of raising and lowering operators, and familiarity with the NRQM anti/symmetry requirements of multiple identical-particle fermion/boson systems. It is deliberately repetitive in some ways, and progresses slowly from the NRQM wave function of a single particle to the quantum field state of the universe.

Overview: Why Quantum Fields?

So far, we've worked entirely in the relativistic single-particle wave theory. Though this is "relativistic quantum mechanics", it uses the unquantized wave-function. The failing of the wave-mechanical interpretation of the relativistic Klein-Gordon and Dirac equations is that it (like NRQM) describes a single, eternal particle. However, particles are observed to be created and annihilated, and this also implies effects on scattering processes from intermediate (unobserved) particles. A complete theory of physics must explain and quantify such processes.

> Quantum Field Theory extends the concept of a quantum state
> from a wave-function of definite particle number
> to a quantum field state that allows for an uncertain particle number.

QFT is more general than NRQM: all wave-functions have a quantized field representation, but not all quantum field states can be represented as wave-functions. We will show that a QFT state can be thought of as a *set* of wave-functions, one for each combination of modes and excitations. Normalization is usually such that the sum of the norms of the set of wave-functions is 1.

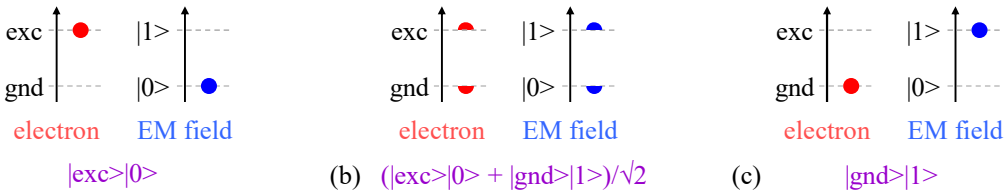

Figure 5.1 (a) An excited atom with 0 photons. (b) After one half-life, there is a superposition of excited and decayed atom, with an entangled superposition of 0 and 1 photons. (c) A long time later, there is (almost certainly) a decayed atom with 1 photon.

For example, consider an excited atom in a large box (Figure 5.1a). We then wait a while, and peek inside to see if the atom has decayed and radiated a photon. During the wait, the quantum state of the system is an evolving entangled atom state and EM field state:

$$|\Psi(t)\rangle = a(t)\underbrace{|exc\rangle}_{\text{atom}}\underbrace{|N=0\rangle}_{\text{EM field}} + b(t)\underbrace{|gnd\rangle}_{\text{atom}}\underbrace{|N=1\rangle}_{\text{EM field}}$$

$$\textit{where} \quad N \equiv \#\,\text{photons}; \qquad (5.1)$$

$$a(0)=1, b(0)=0, \quad a(t\to\infty)=0, b(t\to\infty)=1.$$

Over time, $a(t)$ decreases, while $b(t)$ increases, i.e. the probability of finding an excited atom goes down, and the probability of finding a radiated photon goes up. When we observe the quantum state, we see either the atom is still excited with no photon, or it is in its ground state with a photon in the box. After one half-life, the system state might be:

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}\underbrace{|exc\rangle}_{\text{atom}}\underbrace{|N=0\rangle}_{\text{EM field}} + \frac{1}{\sqrt{2}}\underbrace{|gnd\rangle}_{\text{atom}}\underbrace{|N=1\rangle}_{\text{EM field}}.$$

For $0 < t < \infty$, the photon number of the EM field state is uncertain: it might be measured as either 0 or 1. The *average* photon number $\langle\psi|\hat{N}|\psi\rangle$ in such a state is a fraction between 0 and 1. For longer waits, the probability of seeing the atom excited goes to 0, and seeing it decayed goes to 1; consequently, the probability of seeing a photon rises to 1.

We can use ordinary QM to describe the electron state over time, because there is always exactly one electron. But to have a superposition of EM field states with different photon numbers, the EM field *must* be quantized. A stronger example of the need for quantization is the Lamb shift, where the atom's *stationary* state includes both no-photon and photon components [Be&Sa 1957 p97b, there should be a better reference??]. Such a state can only be described with a quantized EM field. We're used to saying atoms are made of electrons, protons, and neutrons. But we should also say they're made, in part, of photons.

Fundamental States and Creationism

A **quantum field** is a physical field that can be excited to discrete levels, which we call **particles**. For example, photons are the particles (excitations) of the EM field. To build a bridge from NRQM to QFT, we first describe field states of definite particle number. As we saw in (5.1), more general states have uncertain particle number; we describe those after the states of definite particle number.

Boson States

Consider a quantum mechanical system of countably infinite stationary states ψ_0, ψ_1, ... This might be an atom of bound states, an SHO, any other binding potential, or states of any abstract system. It is usually written in Dirac notation as $|N\rangle \equiv \psi_N$. This is called a number basis (or Fock basis). The ψ_N are orthogonal, because they are stationary, and therefore eigenstates of the hamiltonian. We choose them to be normalized, as well.

Recall that a linear operator can be defined by its action on every vector of a basis. Therefore, for *any* orthonormal set of discrete states $|N\rangle$, $N = 0, 1, ...$, we can always define operators $\hat{a}^\dagger$ and $\hat{a}$ such that:

$$\hat{a}|N\rangle = \sqrt{N}\,|N-1\rangle, \quad \hat{a}|0\rangle = \mathbf{0}_v, \quad \hat{a}^\dagger|N\rangle = \sqrt{N+1}\,|N+1\rangle \qquad [\text{cf Bay1990 19-1p411}] \qquad (5.2)$$

$$\textit{where} \quad N = \text{integer} \geq 0.$$

Recall that the zero vector $\mathbf{0}_v$ (aka "null ket") is not $|0\rangle$. There is no Dirac notation (no ket notation) for $\mathbf{0}_v$, and its magnitude is zero. In contrast, $|0\rangle$ is the normalized state vector of the ground state, with unit magnitude. $\hat{a}^\dagger$ and $\hat{a}$ are raising and lowering operators; in QFT, they are often called creation and annihilation operators, but this can be misleading. From these general definitions of $\hat{a}^\dagger$ and $\hat{a}$, we here easily show that they satisfy the familiar relations [Bay 1990 p412]:

$$\hat{N} = \hat{a}^\dagger \hat{a}: \qquad \hat{a}^\dagger \hat{a}\left|N\right\rangle = \left(\sqrt{N}\sqrt{N}\right)\left|N\right\rangle = N\left|N\right\rangle \qquad (N \geq 1)$$

$$\hat{a}^\dagger \hat{a}\left|0\right\rangle = \hat{a}^\dagger \mathbf{0}_\mathbf{v} = \mathbf{0}_\mathbf{v} \qquad (N = 0)$$

$$\left[\hat{a}, \hat{a}^\dagger\right] = \mathbf{1}_{op}: \qquad \left(\hat{a}\hat{a}^\dagger - \hat{a}^\dagger \hat{a}\right)\left|N\right\rangle = \left(N + 1 - N\right)\left|N\right\rangle = \mathbf{1}_{op}\left|N\right\rangle.$$

Such operators are indeed hermitian conjugates (proved below). Thus:

> The concept of raising/lowering/creation/annihilation is applicable to *any* countably infinite orthonormal set of basis states.

It has *nothing* to do with harmonic oscillators, despite having been first discovered by Dirac in the NRQM SHO from a linear combination of position and momentum operators. (However, when we later introduce the field operators, we *do* rely on the ansatz (guess) that the field is a linear combination of raising and lowering operators, as in the SHO.)

Note carefully that $\hat{a}^\dagger$ and $\hat{a}$ are *mathematical* operators that "create" state vectors (kets) representing various quantum field states, but the *physical process* of creating/annihilating an actual particle is more complicated, and involves time-evolution. QFT uses $\hat{a}^\dagger$ and $\hat{a}$ to help describe that physical process.

Furthermore, recall the definition of "adjoint operator", and that we can think of *any* operator as *either* acting to the right (most common), or acting to the left (less common). We choose whether an operator acts right or left at our own convenience. Therefore, since $\hat{a}$ mathematically removes a particle from a ket when acting to the right, it must *add* a particle to a bra when acting to the left:

$$\left\langle 0\right|\hat{a} = \left\langle 1\right|, \qquad \text{and} \qquad \left\langle N\right|\hat{a} = \sqrt{N+1}\left\langle N+1\right|.$$

Similarly, $\hat{a}^\dagger$ mathematically adds a particle when acting to the right, or removes a particle when acting to the left:

$$\left\langle 1\right|\hat{a}^\dagger = \left\langle 0\right|, \qquad \text{and} \qquad \left\langle N\right|\hat{a}^\dagger = \sqrt{N}\left\langle N-1\right|.$$

All the states $N > 0$ are called **excitations** of the system. For a quantum field, the excitations are particles: an excitation of the electron field is an electron (or a positron; we'll get there). An excitation of the EM field is a photon, etc. Each particle type has its own field: electron, photon, muon, tau, up-quark, down-quark, etc. For a quantum field, the vacuum state |0> is an abstract ket, and cannot be written as a function of space-time (the way a wave-function can), but it *is* normalized. These properties are similar to those of a spin ket, such as |↑>.

Position basis: In practical QFT, we usually work in the momentum basis, but as with most expositions, we describe first the position basis, to connect better with NRQM and with our intuitions of space and time. In each quantized field (i.e., for each kind of particle), in the position basis, *we have a quantum system at each point in space*: each point has its own raising/lowering operators, and its own infinite set of excited states (recall we're describing bosons here). (We might imagine this is plausible because the wave-function at each point "oscillates" with angular frequency $\omega = E/\hbar$, and then we imagine these oscillations are quantized.) Therefore we have a continuously infinite set of $\hat{a}^\dagger$ and $\hat{a}$, labeled $\hat{a}_\mathbf{x}^\dagger$ and $\hat{a}_\mathbf{x}$, or $\hat{a}^\dagger(\mathbf{x})$ and $\hat{a}(\mathbf{x})$, and at each point $\mathbf{x}$ we have a *discrete* set of excitation basis states, |0>, |1>, |2>, …, without limit for bosons. The vacuum (at a point $\mathbf{x}$) is the ground state of no particles at position $\mathbf{x}$, $|0_\mathbf{x}>$. The quantum field state of a particle localized at position $\mathbf{x}$ is equivalently denoted several ways:

$$\left|\mathbf{x}\right\rangle = \hat{a}_\mathbf{x}^\dagger\left|0\right\rangle \equiv \hat{a}^\dagger(\mathbf{x})\left|0\right\rangle \equiv \left|N_\mathbf{x} = 1\right\rangle \equiv \left|1_\mathbf{x}\right\rangle \quad \leftrightarrow \quad \text{NRQM } \delta\text{-function located at } \mathbf{x}.$$

This corresponds to an NRQM wave-function of a δ-function located at $\mathbf{x}$. For bosons, N can be any whole number, representing N particles at the same point $\mathbf{x}$. (For fermions, all N are 0 or 1 (the state space at each point is only 2 dimensional).) In scattering, we rarely consider $N \geq 2$, but quantum optics frequently uses $N \geq 2$:

$$\left|N_\mathbf{x}\right\rangle = \frac{1}{\sqrt{N_\mathbf{x}!}}\left(\hat{a}_\mathbf{x}^\dagger\right)^{N_\mathbf{x}}\left|0\right\rangle, \qquad N_\mathbf{x} = 0, 1, 2, ..., \qquad \text{so} \qquad \left\langle N_\mathbf{x}\right|\left.N_\mathbf{x}\right\rangle = 1. \tag{5.3}$$

Note carefully the distinction between a *basis* state and a *quantum field* state (similar to [Dirac 1958 p230]): a **basis state** is a physical quantum state that, at some given time, may or may not be occupied by one or more particles. It's a placeholder into which we may put particles. Usually, we choose basis states that are fixed; however, the occupancy of those states may vary. In the example of a decaying atom, there is a ground state and an excited state of the electron. We say these two basis states "exist" at all times, but are not always occupied. The electron may be in either one of those states, or in a superposition of the two. And the occupancy changes over time: the electron spontaneously migrates from the excited state to the ground state.

A basis state is sometimes called a **mode**, or **single-particle state**, though for bosons, it may be occupied by any number of particles. Such basis states (modes) are usually quantized by boundary conditions on the wave equation, just as in classical physics, and in NRQM. For example, EM cavity modes are discrete. A quantum particle in a box has discrete energies. One might call this "first quantization." In contrast, infinite plane waves are continuous. Even though they are unphysical, they can be used as a mathematical basis for computation.

For multiple particle types, such as an atomic electron and a photon in our example, a **quantum field state** is generally a combination (tensor product) of occupations of the various particle basis states, together with a single complex amplitude.

Since each mode in a quantum field state has its own excitation level, the vacuum field state |0> is a shorthand for *all* modes having no excitations:

$$|0\rangle \equiv \underbrace{|0,0,0\ldots\rangle}_{\text{all modes}} .$$

In the position basis we've used so far, the basis states are of a particle localized to a position **x**. The quantum field state $|N_{\mathbf{x}}\rangle$ means there are $N_{\mathbf{x}}$ particles in the physical, single-particle, basis state at **x**. As in statistical mechanics, the N are called **occupation numbers**, because they tell how many particles occupy the given physical state. One might call the discrete, quantized occupation states "second quantization."

In general, a basis set may be continuous or discrete. Particles in free space have a continuous basis, such as position **x**, or momentum **p**. In contrast, excitations of an EM cavity have a discrete basis: the resonant modes of the EM field. It is common to derive results in a finite box (a discrete basis), and then take the **continuum limit**: the limit as the box volume goes to infinity, and the discrete basis becomes continuous.

A quantum field has a quantum system at each point **x**, and therefore **x** and **x** + d**x** denote two quantum systems. However, being "adjacent," they are coupled (not independent). Over all space, the quantum field (just like a classical field) is a system of infinite degrees of freedom, one at each point **x**.

Proof that $\hat{a}^\dagger$ and $\hat{a}$ are adjoints: To prove that $\hat{a}^\dagger$ and $\hat{a}$ are hermitian conjugates (adjoints), we use the definition of adjoint: $\hat{a}^\dagger$, when acting to the left, produces a bra from a bra the same way that $\hat{a}$ (acting to the *right*) produces a ket from a ket:

$$\langle N|\hat{a}^\dagger \equiv \langle \hat{a}N| = \left(\hat{a}|N\rangle\right)^{DC} = \sqrt{N}\langle N-1| \qquad \textit{where} \qquad DC \equiv \text{dual conjugate:} |\psi\rangle^{DC} \equiv \langle\psi| .$$

A common way to prove the equality of two operators is to show that they have identical inner products between all pairs of basis kets. Does the above equation insure that? From the above adjoint definition of $\hat{a}^\dagger$, for every pair of basis kets |M> and |N>:

$$\langle N|\hat{a}^\dagger|M\rangle = \sqrt{N}\langle N-1|M\rangle = \begin{cases} \sqrt{N} & \text{when } N-1=M \\ 0 & \text{otherwise} \end{cases} .$$

For comparison, our original definition (5.2) gives:

$$\langle N|\hat{a}^\dagger|M\rangle = \sqrt{M+1}\langle N|M+1\rangle = \begin{cases} \sqrt{N} & \text{when } N=M+1 \\ 0 & \text{otherwise} \end{cases} .$$

These two results are identical, proving our definition (5.2) of $\hat{a}^\dagger$ is indeed the adjoint of $\hat{a}$.

We discuss the time dependence of $\hat{a}^\dagger$ and $\hat{a}$ in a later chapter.

Momentum Basis

In the momentum basis, our single-particle states have definite momentum, rather than definite position. Thus, we define operators to raise/lower field states of single-particle momentum eigenstates. All the above results carry over with the simple substitution $\mathbf{x} \rightarrow \mathbf{p}$. The field states $|N_\mathbf{p}\rangle$ tell how many particles are in the physical state of momentum $\mathbf{p}$. A field state of a single particle with momentum $\mathbf{p}$ can be equivalently written several ways:

$$|\mathbf{p}\rangle = \hat{a}_\mathbf{p}^\dagger|0\rangle = |N_\mathbf{p} = 1\rangle \equiv |1_\mathbf{p}\rangle \qquad \textit{where} \quad |0\rangle \equiv \text{vacuum};\ \hat{a}_\mathbf{p}^\dagger \text{ raises the momentum state excitation .}$$

Note the difference between $\hat{a}_\mathbf{p}^\dagger$ and $\hat{a}_\mathbf{x}^\dagger$: $\hat{a}_\mathbf{p}^\dagger$ creates a field state of a particle of definite momentum $\mathbf{p}$, and $\hat{a}_\mathbf{x}^\dagger$ creates a field state of a particle with definite position $\mathbf{x}$. The physical (basis) state $|\mathbf{p}\rangle$ covers all points in space; $|\mathbf{x}\rangle$ covers all points in momentum-space. When written in the position basis, the physical state $|\mathbf{p}\rangle$ has the same spatial dependence as the NRQM wave-function, $\exp(i\mathbf{p}\bullet\mathbf{x})$:

$$|\mathbf{p}\rangle = \int_\infty \exp(i\mathbf{p}\bullet\mathbf{x})|\mathbf{x}\rangle\, d^3x\ .$$

Boson Commutation Relations

For *any* discrete set of states, $|0\rangle$, $|1\rangle$, $|2\rangle$, …, we defined raising and lowering operators, $\hat{a}^\dagger$ and $\hat{a}$. The states might be position eigenstates, momentum eigenstates, or any other states (of any kind of system). From those raising and lower definitions, we easily derive the commutation relations by acting on an arbitrary state $|N\rangle$:

$$\begin{aligned} \left[\hat{a}, \hat{a}^\dagger\right]|N\rangle \equiv \left(\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\right)|N\rangle &= \left(\sqrt{N+1}\sqrt{N+1} - \sqrt{N}\sqrt{N}\right)|N\rangle = \mathbf{1}_{op}|N\rangle \\ \Rightarrow \qquad \left[\hat{a}, \hat{a}^\dagger\right] &= \mathbf{1}_{op} \qquad \text{(the identity operator).} \end{aligned} \tag{5.4}$$

Why do we care about commutation relations? Because they capture a lot of major phenomena in a few simple relations. And they are often useful computational aids. We will use them extensively.

We will say more about commutators when we discuss multi-particle states.

Fermion States

Experiment shows that fermions satisfy the exclusion principle, so any given single-particle state can have only occupation numbers of $N = 0$ or 1. This requires a slightly different raising/lowering scheme than bosons. We still have a vacuum state, and a state of 1 particle: $|0\rangle$ and $|1\rangle$. But we must insure that any computation that would lead to a second excitation, $|2\rangle$, has zero probability. An easy way to do that is to define our raising operator differently than bosons. (Fermion operators are often written with b rather than a):

$$\hat{b}^\dagger|0\rangle \equiv |1\rangle \ \text{ (as with bosons)}, \qquad \hat{b}^\dagger|1\rangle \equiv \mathbf{0}_\mathbf{v}\ .$$

Probabilities are computed from inner products, and any inner product with $\mathbf{0}_\mathbf{v}$ produces zero, so the probability of creating any state higher than $|1\rangle$ is zero.

Fermion Commutation Relations

This $\hat{b}^\dagger$ modification preserves the number operator as $\hat{b}^\dagger\hat{b} = \hat{N}$, but it breaks our commutator:

$$\left[\hat{b}, \hat{b}^\dagger\right]|1\rangle = \left(\hat{b}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\right)|1\rangle = (0-1)|1\rangle \neq |1\rangle \quad \text{so} \qquad \left[\hat{b}, \hat{b}^\dagger\right] \neq \mathbf{1}_{op}\ . \tag{5.5}$$

Instead of $\hat{a}\hat{a}^\dagger = N+1$, we have $\hat{b}\hat{b}^\dagger = 1-N$. The constant operator we can construct is the *anti*-commutator:

$$\hat{b}\hat{b}^\dagger + \hat{b}^\dagger\hat{b} \equiv \left\{\hat{b}, \hat{b}^\dagger\right\} = \mathbf{1}_{op} .$$

As we've already seen with the Dirac equation,
anti-commutators are at the heart of fermions.

We will say more about commutators when we discuss multi-particle states.

Comments on Quantum Field States

Some people claim there are no particles, only fields. We think that's unfounded. Of course particles exist: we observe them in experiments all the time. We represent them mathematically as excitations of fields. There's no conflict in that.

Wave-functions from Field States: There and Back Again

When we graduate from NRQM to QFT, we step away somewhat from wave-functions toward field states and field operators. It is important to understand the relationship between the above field states and NRQM wave-functions, because this illustrates how QFT reduces to NRQM in the low-energy limit. In the NR limit (which precludes any particle creation/annihilation, including photons), the single-particle NRQM gives the same results as the quantum field theory. So what is the relationship between a single-particle wave-function and a quantum field excitation? We now examine this in several ways, to provide a solid foundation for QFT. (Several sources mention the connection between quantum field states and NRQM wave-functions, though in much less detail [Bay p422], [Sak 1967 p147b], [P&E p24b], and [Wei 2013 p. xvi top].)

We use the terms "probability density" and "particle density" interchangeably.

Single-Particle Wave-function From Raising/Lowering

We start wave-function/QFT relations with field states of 1 particle, and later advance to multi-particle states. First, consider a single fermion quantum field state localized at position **x**. We drop spin for simplicity. Fermion raising/lowering operators are often written as $\hat{b}^\dagger(\mathbf{x})$ and $\hat{b}(\mathbf{x})$:

$$|\phi\rangle = \hat{b}^\dagger(\mathbf{x})|0\rangle = |1_\mathbf{x}\rangle \qquad \textit{where} \quad \hat{b}^\dagger(\mathbf{x}) \equiv \hat{b}^\dagger_\mathbf{x} \quad \text{(single particle localized at } \mathbf{x}\text{)}.$$

The NRQM wave-function for this is a sharp spike at **x**, and zero everywhere else, i.e. a δ-function at **x**. Now consider a single fermion in a superposition of two locations:

$$|\phi\rangle = w_1\hat{b}^\dagger(\mathbf{x}_1)|0\rangle + w_2\hat{b}^\dagger(\mathbf{x}_2)|0\rangle \qquad \textit{where} \quad w_1, w_2 \text{ are the amplitudes of the superposition .}$$

The w's are complex amplitudes, or *weights*. Now generalize to a single fermion at a superposition of k locations:

$$|\phi\rangle = \sum_{j=1}^{k} w_j\hat{b}^\dagger(\mathbf{x}_j)|0\rangle .$$

Generalize further to a single fermion in a continuously infinite superposition of all locations, where each location has an amplitude $w(\mathbf{x})$. In other words, take the continuum limit of the above, with $w(\,)$ as a complex "weight" function:

$$|\phi\rangle = \int_\infty d^3x\, w(\mathbf{x})\hat{b}^\dagger(\mathbf{x})|0\rangle \;\Rightarrow \qquad \phi(\mathbf{x}) = w(\mathbf{x}) . \tag{5.6}$$

$w(\mathbf{x})$ is the complex amplitude for the particle to be found at **x**,
which by definition, is the NRQM wave-function for the particle.

For single-particle field states, the wave-function is simply the weight function $w(\mathbf{x})$. We show later that this does *not* hold for higher n-particle states.

If there is definitely one particle in the field state, it must be that:

$$\int_\infty d^3x\,|w(\mathbf{x})|^2 = 1\,.$$

i.e., w (and therefore ϕ) is normalized.

Eq. (5.6) provides a simple way to construct the single-particle NRQM wave-function $\psi(\mathbf{x})$ from the QFT state $|\phi\rangle$: to find $\psi(\mathbf{x})$, we simply couple $|\phi\rangle$ to the vacuum with a lowering operator at $\mathbf{x}$:

$$\psi(\mathbf{x}) = \langle 0|\hat{b}(\mathbf{x})|\phi\rangle \qquad \text{because } \hat{b}(\mathbf{x})\hat{b}^\dagger(\mathbf{x})|0\rangle = |0\rangle, \quad \text{and for } \mathbf{x}\neq\mathbf{x}',\ \hat{b}(\mathbf{x})\hat{b}^\dagger(\mathbf{x}')|0\rangle = \mathbf{0}_\mathbf{v}\,.$$

Then:

$$\langle 0|\hat{b}(\mathbf{x})|\phi\rangle = w(\mathbf{x})\underbrace{\langle 0|0\rangle}_{1},$$

as desired. This means we can make the following operator equivalence:

$$\hat{b}(\mathbf{x})\hat{b}^\dagger(\mathbf{x})|0\rangle = |0\rangle, \quad \text{and for } \mathbf{x}\neq\mathbf{x}',\ b(\mathbf{x})\hat{b}^\dagger(\mathbf{x}')|0\rangle = \mathbf{0}_\mathbf{v} \qquad \Rightarrow \qquad b(\mathbf{x})\hat{b}^\dagger(\mathbf{x}') = \delta^3(\mathbf{x}-\mathbf{x}')\,.$$

Same for bosons with $\hat{b}\to\hat{a}$.

Example: Atomic Dipole Radiation: A Single-Particle State

To illustrate some of these concepts, we give here a simplified example of a single-photon EM field state. After an excited atom decays and radiates, the emitted photon is in a single-particle state, though it is in a superposition of locations, of momenta, and of energy. A dipole-radiated field state is a simple and realistic example (Figure 5.2).

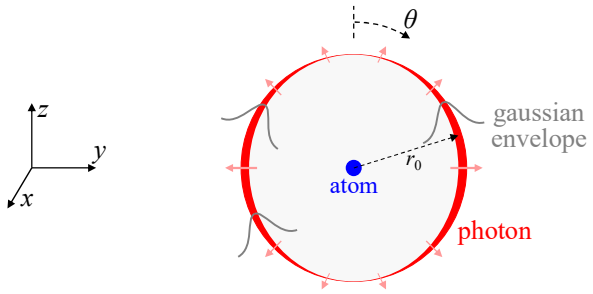

Figure 5.2 A single-photon, dipole radiated EM field state. The wave packet is an expanding thick spherical shell. The thickness of the red circle suggests the relative EM field *intensity* in various directions, whereas the spatial thickness of the actual radiation shell is the same in all directions.

Classically, and quantumly, the radiation field spreads away from the atom in a thick shell of spherical waves, so we use spherical coordinates. Dipole radiation has an angular amplitude function of $Y_{10}(\theta, \phi) \propto \sin\theta$.

For simplicity, we ignore polarization, so we can treat the EM field as a scalar boson. The shell is a wave packet, which we approximate as a gaussian envelope along every radius. Note that applying an envelope to a wave of definite momentum turns it into a superposition of momenta. Since the energy of a photon is $E = cp$, a superposition of momenta is also a superposition of energy. This is *not* a stationary state: it propagates radially outward.

We consider a time when the radial wave packet is centered at a distance r_0. The wave-function is axially symmetric (no ϕ dependence). In the position basis, the weight function is then ($\hbar = 1$):

$$w(r,\theta,\phi) = R(r)Y_{10}(\theta,\phi) = A \underbrace{\frac{\exp(ikr)}{r}}_{\text{spherical wave}} \underbrace{\exp\left[(r-r_0)^2 / 2\sigma^2\right]}_{\text{wave packet envelope}} \sin\theta$$

where $k \equiv$ central momentum

$A \equiv$ normalization such that $\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \int_0^{\infty} r^2 dr \left|w(r,\theta,\phi)\right|^2 = 1$.

The radiation intensity is 0 along $\pm z$, and maximum in the x-y plane, a characteristic of dipole radiation.

We now construct the EM field state in the position basis, in spherical coordinates.

$$\begin{aligned}\left|\phi_{EM}\right\rangle &= \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \int_0^{\infty} r^2 dr \, w(r,\theta,\phi)\left|r,\theta,\phi\right\rangle \\ &= \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \int_0^{\infty} r^2 dr \, A\frac{\exp(ikr)}{r}\exp\left[(r-r_0)^2 / 2\sigma^2\right]\cos\theta\left|r,\theta,\phi\right\rangle.\end{aligned}$$

$|r, \theta, \phi\rangle$ is a single particle excitation ($N = 1$) at the point (r, θ, ϕ), just like $|x, y, z\rangle$ or $|\mathbf{x}\rangle$ is.

We can write the total number operator as a sum over all positions:

$$\hat{N} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \int_0^{\infty} r^2 dr \, \hat{a}^{\dagger}(r,\theta,\phi)\hat{a}(r,\theta,\phi).$$

$|\phi_{EM}\rangle$ is an eigenstate of the total number operator, with eigenvalue 1, because it is a superposition of single-photon position states. Thus, there is exactly one photon in this field state (we omit the integral proof):

$$\hat{N}\left|\phi_{EM}\right\rangle = \left|\phi_{EM}\right\rangle.$$

Of course, the *average* particle number is also 1: $\left\langle\phi_{EM}\right|\hat{N}\left|\phi_{EM}\right\rangle = 1$.

A more accurate description of the photon would include polarization (and so would be a vector spherical harmonic function of space).

Sometime people object to using the position basis for photons, arguing that a photon can't physically be localized. But this is not a problem: the *mathematical* basis states we use for computations need not be composed of physically realizable states. No particle can be physically localized to a point, but we use the x-basis all the time. And both classically and quantumly, we use a Green function for EM propagation, which starts with an EM field localized to a point, and even includes spatial *derivatives*, all legitimately. Similarly, there is no such thing as infinite plane waves (momentum basis), but we use them profitably, as well.

Multiple Identical-Particle Wave-function From Raising/Lowering

We now construct *multiple* identical-particle NRQM wave-functions and quantized field states, in the position basis. For now, we consider only singly-excited physical states. Fermions can be at most singly-excited. We show how multiparticle states introduce symmetry requirements for wave-functions. We mostly ignore here any normalization, to be considered soon.

Definition: We define a **configuration** of an n-particle system as an *unordered* set of n basis states, e.g. $\{\mathbf{x}_1, \mathbf{x}_2\}$ is a configuration (in the position basis: locations of particles) of a 2-particle system. It is the same as the configuration $\{\mathbf{x}_2, \mathbf{x}_1\}$, because order doesn't matter. We focus on 2-particle states, which generalize simply to arbitrary n-particle states.

We sometimes write multiparticle field states as kets with multiple parameters, one for each occupied mode. For example, for singly-occupied localized 2-particle states:

$\left|\mathbf{x}_a, \mathbf{x}_b\right\rangle$ in the position basis;

$\left|\mathbf{p}_a, \mathbf{p}_b\right\rangle$ in the momentum basis.

Multiple identical particle states such as $|\mathbf{p}_a, \mathbf{p}_b\rangle$ are common input or output states for simple scattering events, such as electron-electron scattering.

Be careful with the factorials: there are at least 2 different factorials in this chapter that look similar, but are different: (5.3), and (5.17).

Fermion Multi-particle States

Consider a quantum field state of two identical fermions, localized at $\mathbf{x}_a$ and $\mathbf{x}_b$, $\mathbf{x}_a \neq \mathbf{x}_b$:

$$|\phi\rangle = |\mathbf{x}_a, \mathbf{x}_b\rangle \equiv \hat{b}^\dagger(\mathbf{x}_b)\hat{b}^\dagger(\mathbf{x}_a)|0\rangle \quad \Rightarrow \quad \langle \mathbf{x}_a, \mathbf{x}_b | \mathbf{x}_a, \mathbf{x}_b \rangle = 1 \tag{5.7}$$

Let us now construct the 2-particle NRQM wave-function $\phi(\mathbf{x}_1, \mathbf{x}_2)$ from the QFT state $|\phi\rangle$. From NRQM, it must be anti-symmetric, i.e. $\phi(\mathbf{x}_1, \mathbf{x}_2) = -\phi(\mathbf{x}_2, \mathbf{x}_1)$. Note that there is no concept of which particle is which: they are identical. There is no "first" particle or "second" particle; there are simply two *identical* particles. However, the anti-symmetry requirement says there *is* a distinction between the first *argument* to ϕ and the second. We associate the first argument with the first raising operator in (5.7), i.e. at $\mathbf{x}_a$. We associate the second argument with the second raising operator, at $\mathbf{x}_b$. Note that in (5.7) the raising operators are *written* in the reverse order as the labels $\mathbf{x}_a$ and $\mathbf{x}_b$ appear in the ket, but they *act* in the *same* order: $\mathbf{x}_a$ first [Bay 19-26 p417].

> Many references refer to wave-function anti-symmetry under "particle" exchange. That's misleading, because both particles are identical, and there is no meaning to "exchanging" them. The wave-function is anti-symmetric under *label* exchange: $\mathbf{x}_1 \leftrightarrow \mathbf{x}_2$. This anti-symmetry has important observable consequences.

Since these example particles are localized, we use δ-functions for the wave-function. We choose the phase to be 1 when the arguments to $\phi(\bullet)$ are in the same order as the particles are mathematically (not physically) created in (5.7). Then we couple the QFT state to the vacuum with lowering operators, similar to the single-particle case. But to get the phase we want, we must order the lowering operators to act in reverse of the raising in (5.7):

$$\begin{aligned}\phi(\mathbf{x}_1, \mathbf{x}_2) &= \underbrace{\langle 0|\hat{b}(\mathbf{x}_1)\hat{b}(\mathbf{x}_2)}_{\langle \mathbf{x}_1, \mathbf{x}_2|}|\phi\rangle = \langle 0|\hat{b}(\mathbf{x}_1)\underbrace{\hat{b}(\mathbf{x}_2)\hat{b}^\dagger(\mathbf{x}_b)}_{\delta^3(\mathbf{x}_2-\mathbf{x}_b)}\hat{b}^\dagger(\mathbf{x}_a)|0\rangle \\ &= \delta^3(\mathbf{x}_2-\mathbf{x}_b)\langle 0|\underbrace{\hat{b}(\mathbf{x}_1)\hat{b}^\dagger(\mathbf{x}_a)}_{\delta^3(\mathbf{x}_1-\mathbf{x}_a)}|0\rangle = +\delta^3(\mathbf{x}_2-\mathbf{x}_b)\delta^3(\mathbf{x}_1-\mathbf{x}_a)\cancel{\langle 0|0\rangle} \quad + more\,.\end{aligned} \tag{5.8}$$

(We'll complete this wave-function in a minute.) This is a δ-function 2-particle wave-function, as expected. Note that if $\mathbf{x}_1 = \mathbf{x}_a$ and $\mathbf{x}_2 = \mathbf{x}_b$, then we annihilated in the order that avoids any commuting of operators, so the phase is +1.

What happens when we reverse the order of arguments to ϕ?

$$\phi(\mathbf{x}_1 = \mathbf{x}_b, \mathbf{x}_2 = \mathbf{x}_a) = \langle 0|\hat{b}(\mathbf{x}_b)\hat{b}(\mathbf{x}_a)|\phi\rangle = \langle 0|\hat{b}(\mathbf{x}_b)\underbrace{\hat{b}(\mathbf{x}_a)\hat{b}^\dagger(\mathbf{x}_b)}_{?}\hat{b}^\dagger(\mathbf{x}_a)|0\rangle\,. \tag{5.9}$$

We cannot eliminate the indicated operators because they act at different points $\mathbf{x}$. The result is not $\mathbf{0}_V$ though, because $\hat{b}(\mathbf{x}_a)$ acts on a particle at $\mathbf{x}_a$ anywhere to the right. But we can evaluate the RHS by commuting any two adjacent operators. To make the wave-function anti-symmetric, we must *define* this commutation (and all others between any two raising/lowering operators at different points) as including a negative sign for fermions:

$$\begin{aligned}\text{For } \mathbf{x}_a \neq \mathbf{x}_b: \quad & \hat{b}(\mathbf{x}_a)\hat{b}(\mathbf{x}_b) \equiv -\hat{b}(\mathbf{x}_b)\hat{b}(\mathbf{x}_a), \qquad \text{and} \\ & \hat{b}^\dagger(\mathbf{x}_a)\hat{b}^\dagger(\mathbf{x}_b) \equiv -\hat{b}^\dagger(\mathbf{x}_b)\hat{b}^\dagger(\mathbf{x}_a), \qquad \hat{b}(\mathbf{x}_a)\hat{b}^\dagger(\mathbf{x}_b) \equiv -\hat{b}^\dagger(\mathbf{x}_b)\hat{b}(\mathbf{x}_a)\,.\end{aligned} \tag{5.10}$$

For example, commuting the indicated operators in (5.9) above gives:

$$\phi(\mathbf{x}_1 = \mathbf{x}_b, \mathbf{x}_2 = \mathbf{x}_a) = -\langle 0 | \underbrace{\hat{b}(\mathbf{x}_b)\hat{b}^\dagger(\mathbf{x}_b)}_{\delta^3(\mathbf{x}-\mathbf{x}_b)} \underbrace{\hat{b}(\mathbf{x}_a)\hat{b}^\dagger(\mathbf{x}_a)}_{\delta^3(\mathbf{x}-\mathbf{x}_a)} | 0 \rangle$$

$$= -\delta^3(\mathbf{x}_a - \mathbf{x}_a)\delta^3(\mathbf{x}_b - \mathbf{x}_b)\cancel{\langle 0 | 0 \rangle} .$$

as expected. Our defining QFT state (5.7) lists the particle configuration (locations $\mathbf{x}_a$ and $\mathbf{x}_b$) just once, but using the anti-commutation relations (5.10), the corresponding wave-function (5.11) is *automatically* anti-symmetric. Thus, for *any* $\mathbf{x}_1$, $\mathbf{x}_2$, the wave-function is (ignoring normalization):

$$\phi(\mathbf{x}_1, \mathbf{x}_2) = \langle 0 | \hat{b}(\mathbf{x}_2)\hat{b}(\mathbf{x}_1) | \phi \rangle = \delta^3(\mathbf{x}_1 - \mathbf{x}_a)\delta^3(\mathbf{x}_2 - \mathbf{x}_b) - \delta^3(\mathbf{x}_2 - \mathbf{x}_a)\delta^3(\mathbf{x}_1 - \mathbf{x}_b) . \tag{5.11}$$

We don't need to explicitly anti-symmetrize it, as in NRQM. In other words:

> Anti-commuting raising/lowering operators imply anti-symmetric wave-functions, and vice versa. The two conditions are equivalent.

Note that $\phi(\mathbf{x}_1, \mathbf{x}_2) = 0$ for all $(\mathbf{x}_1, \mathbf{x}_2)$ except $(\mathbf{x}_a, \mathbf{x}_b)$ and $(\mathbf{x}_b, \mathbf{x}_a)$.

A similar analysis for bosons shows that wave-functions (for all $N_\mathbf{x} = 0$ or 1, and $\hat{a}^\dagger$ and $\hat{a}$ *commute*) are automatically *symmetrized*. Thus in QFT, fermion and boson symmetrization for wave-functions is *built into* the theory by the commutation relations of the raising and lowering operators.

Note again the denotation of the order of $\mathbf{x}_a$ and $\mathbf{x}_b$ in the ket $|\mathbf{x}_a, \mathbf{x}_b>$, as in (5.7): the raising operators *act* in the same order as $\mathbf{x}_a$ and $\mathbf{x}_b$ appear in the ket:

$$|\mathbf{x}_a, \mathbf{x}_b\rangle \equiv \hat{b}^\dagger(\mathbf{x}_b)\hat{b}^\dagger(\mathbf{x}_a)|0\rangle .$$

This implies the following subtle result of anticommutation (still ignoring normalization):

$$\hat{b}(\mathbf{x}_a)|\mathbf{x}_a, \mathbf{x}_b\rangle = -|\mathbf{x}_b\rangle \qquad \text{and} \qquad \hat{b}(\mathbf{x}_b)|\mathbf{x}_a, \mathbf{x}_b\rangle = +|\mathbf{x}_a\rangle .$$

Furthermore, the states $|\mathbf{x}_a, \mathbf{x}_b>$ and $|\mathbf{x}_b, \mathbf{x}_a>$ are the *same* physical state, but with different quantum phases. They are *not* orthogonal, and in fact:

$$|\mathbf{x}_a, \mathbf{x}_b\rangle = -|\mathbf{x}_b, \mathbf{x}_a\rangle \qquad \text{or} \qquad \langle \mathbf{x}_b, \mathbf{x}_a | \mathbf{x}_a, \mathbf{x}_b \rangle = -1 .$$

For bras, the operators are written in the *same* order as in the ket, so that (similar to kets) the operators *act to the left* in the same order as the labels $\mathbf{x}_a$ and $\mathbf{x}_b$ appear:

$$|\mathbf{x}_a, \mathbf{x}_b\rangle^{DC} \equiv \langle \mathbf{x}_a, \mathbf{x}_b | \equiv \langle 0 | \hat{b}(\mathbf{x}_a)\hat{b}(\mathbf{x}_b) . \tag{5.12}$$

Thus we see from (5.11) that $\phi(\mathbf{x}_1, \mathbf{x}_2) \propto \langle \mathbf{x}_1, \mathbf{x}_2 | \phi \rangle$, similar to the 1-particle case, but with normalization still to-be-determined (TBD).

Continuously Distributed 2-Particle Fermion States

Finally, consider the most realistic state: a two-fermion field state in a continuously infinite superposition (sum) of all possible position pairs, each with complex amplitude $w\{\mathbf{x}_a, \mathbf{x}_b\}$, analogous to (5.6). We use braces for the argument of w to remind us that it is a function of *configurations*. We must be careful, though, because as noted earlier, $|\mathbf{x}_a, \mathbf{x}_b>$ and $|\mathbf{x}_b, \mathbf{x}_a>$ are the *same* physical state (with different phase), so we should include only *one* of them in the superposition. Therefore, our integral over configuration space should include only *unique combinations* of $\mathbf{x}_a$ and $\mathbf{x}_b$, i.e. integrate only over configurations $\{\mathbf{x}_a, \mathbf{x}_b\}$. Our quantum field state is:

$$|\phi\rangle = \iint_{\substack{\text{unique} \\ \text{pairs } \mathbf{x}_a, \mathbf{x}_b}} d^3x_a \, d^3x_b \, w\{\mathbf{x}_a, \mathbf{x}_b\} \hat{b}^\dagger(\mathbf{x}_b)\hat{b}^\dagger(\mathbf{x}_a)|0\rangle . \tag{5.13}$$

This means the probability density of finding the configuration $\{\mathbf{x}_a, \mathbf{x}_b\}$ (a particle at $\mathbf{x}_a$ and one at $\mathbf{x}_b$) is $|w\{\mathbf{x}_a, \mathbf{x}_b\}|^2$. Almost every QFT source I've seen uses this convention (but usually in the momentum basis),

though none of them describe it explicitly. Integrating over "unique pairs" is a mathematical problem, which we address below when discussing normalization.

Note that there is no concept of anti/symmetrization of $w\{\ \}$, since it is defined only over unique configurations: $\{\mathbf{x}_a, \mathbf{x}_b\}$ is the same as $\{\mathbf{x}_b, \mathbf{x}_a\}$. You can't "swap" labels on an unordered set. However, for fermions, we must choose a single ordering for the full set of basis modes, so that given any configuration $\{\mathbf{x}_1, \ldots \mathbf{x}_n\}$, we construct the field state with raising operators in the chosen order of modes. This avoids any minus signs from commuting fermion raising operators.

[Aside: for discrete bases, such an ordering is tedious, but do-able. For continuous bases, this is much harder. We resolve this below, under "normalization".]

To extract the NRQM two-particle wave-function from such a state, we use the same two lowering operators as in (5.11) to couple to the vacuum:

$$\phi(\mathbf{x}_1, \mathbf{x}_2) = A\langle 0|\hat{b}(\mathbf{x}_2)\hat{b}(\mathbf{x}_1)|\phi\rangle = A\langle 0|\hat{b}(\mathbf{x}_2)\hat{b}(\mathbf{x}_1) \underbrace{\iint\limits_{\substack{\text{unique} \\ \text{pairs } \mathbf{x}_a, \mathbf{x}_b}} d^3x_a\, d^3x_b\, w\{\mathbf{x}_a, \mathbf{x}_b\}\hat{b}^\dagger(\mathbf{x}_b)\hat{b}^\dagger(\mathbf{x}_a)|0\rangle}_{\text{QFT state } |\phi\rangle}.$$

The lowering operators can be moved inside the integral to become δ-functions, and pick out the locations where $\mathbf{x}_a = \mathbf{x}_1$ and $\mathbf{x}_b = \mathbf{x}_2$, *or* $\mathbf{x}_b = \mathbf{x}_1$ and $\mathbf{x}_a = \mathbf{x}_2$. Still ignoring normalization, we get the 2-particle wave-function:

$$\phi(\mathbf{x}_1, \mathbf{x}_2) = w(\mathbf{x}_1, \mathbf{x}_2), \qquad \phi(\mathbf{x}_2, \mathbf{x}_1) = -w(\mathbf{x}_1, \mathbf{x}_2) \qquad \text{(unnormalized)}.$$

Again, the antisymmetry of $\phi(\ )$ is built-in by the raising/lowering commutation relations.

Boson 2-particle States

The boson 2-particle single-excitation (per mode, or basis state) case is nearly identical to the fermion case, but with *commuting* raising/lowering operators, which is equivalent to *symmetric* 2-particle wave-functions. Instead of (5.10), we have:

$$\hat{a}(\mathbf{x}_a)\hat{a}(\mathbf{x}_b) \equiv \hat{a}(\mathbf{x}_b)\hat{a}(\mathbf{x}_a),\ \ \hat{a}^\dagger(\mathbf{x}_a)\hat{a}^\dagger(\mathbf{x}_b) \equiv \hat{a}^\dagger(\mathbf{x}_b)\hat{a}^\dagger(\mathbf{x}_a), \qquad \hat{a}(\mathbf{x}_a)\hat{a}^\dagger(\mathbf{x}_b) \equiv \hat{a}^\dagger(\mathbf{x}_b)\hat{a}(\mathbf{x}_a).$$

Therefore, boson wave-functions taken from field states (with all $N_x = 0$ or 1) are automatically symmetrized.

Furthermore, the order of the particles is completely irrelevant, and different orderings have the *same* quantum phase (no minus signs):

$$|\mathbf{x}_a, \mathbf{x}_b\rangle = |\mathbf{x}_b, \mathbf{x}_a\rangle \qquad \text{or} \qquad \langle \mathbf{x}_b, \mathbf{x}_a|\mathbf{x}_a, \mathbf{x}_b\rangle = +1$$
$$\hat{a}(\mathbf{x}_a)|\mathbf{x}_a, \mathbf{x}_b\rangle = +|\mathbf{x}_b\rangle \qquad \text{and} \qquad \hat{a}(\mathbf{x}_b)|\mathbf{x}_a, \mathbf{x}_b\rangle = +|\mathbf{x}_a\rangle.$$

Momentum Basis

In the momentum basis, the total boson or fermion field state is a weighted sum over continuous momentum states, analogous to (5.13). For example, a 2-particle fermion state would be:

$$|\phi\rangle = \iint\limits_{\substack{\text{unique} \\ \text{pairs } \mathbf{x}_a, \mathbf{x}_b}} d^3p_a\, d^3p_b\, w\{\mathbf{p}_a, \mathbf{p}_b\}\hat{b}^\dagger(\mathbf{p}_b)\hat{b}^\dagger(\mathbf{p}_a)|0\rangle.$$

A boson state is the same except $\hat{b}^\dagger \to \hat{a}^\dagger$.

In QFT, fermion and boson symmetrization for wave-functions is *built into* the theory by the commutation relations of the raising and lowering operators.

Total Number Operator

In multiparticle states, we sometimes need the total number of all particles, which is simply the sum of the particles in each configuration. We must sum over all *n*-particle configurations, for all *n*. For example, in a discrete position basis:

$$\hat{N} = \sum_{n=0}^{\infty} \hat{N}\{\mathbf{x}_1, \ldots \mathbf{x}_n\} = 0 + \sum_{\mathbf{x}_a}^{\text{basis-states}} \hat{N}(\mathbf{x}_a) + \sum_{\{\mathbf{x}_a,\mathbf{x}_b\}} \hat{N}\{\mathbf{x}_a,\mathbf{x}_b\} + \sum_{\{\mathbf{x}_a,\mathbf{x}_b,\mathbf{x}_c\}} \hat{N}\{\mathbf{x}_a,\mathbf{x}_b,\mathbf{x}_c\} + \ldots$$

$$\textit{where} \quad \hat{N}\{\mathbf{x}_a,\mathbf{x}_b,\ldots\} = \hat{a}^\dagger(\mathbf{x}_a)\hat{a}(\mathbf{x}_a) + \hat{a}^\dagger(\mathbf{x}_b)\hat{a}(\mathbf{x}_b) + \ldots \qquad \text{and so on .}$$

In the momentum basis, simple replace $\mathbf{x} \to \mathbf{p}$. For example, consider a state where two Majorana neutrinos (in a box) are on their way to annihilate each other (they are their own anti-particles). Then our QFT state is some superposition of the vacuum (i.e., annihilated), and a two-particle ν (fermion) state (pre-annihilation), plus reaction products that we ignore. The average number of particles, noting that the vacuum contributes no particles, is:

$$|\phi\rangle = d|0\rangle + \sum_{\{\mathbf{x}_a,\mathbf{x}_b\}} w\{\mathbf{x}_a,\mathbf{x}_b\}\hat{b}^\dagger(\mathbf{x}_b) + \hat{b}^\dagger(\mathbf{x}_a)|0\rangle \qquad \Rightarrow$$

$$\hat{N}\{\mathbf{x}_a,\mathbf{x}_b\} = \hat{b}^\dagger(\mathbf{x}_b)\hat{b}(\mathbf{x}_b) + \hat{b}^\dagger(\mathbf{x}_a)\hat{b}(\mathbf{x}_a) = 2$$

$$\langle N\rangle = \langle\phi|\hat{N}|\phi\rangle = 0 + \sum_{\{\mathbf{x}_a,\mathbf{x}_b\}} 2\left|w\{\mathbf{x}_a,\mathbf{x}_b\}\right|^2 .$$

Remember that because of the amplitudes in $w\{\ \}$, the average number of particles need not be an integer (as we saw in the example of the decaying atom in a box).

For continuous bases, we have the usual continuum limit. In the position basis:

$$\hat{N} = \sum_{n=0}^{\infty} \hat{N}\{\mathbf{x}_1, \ldots \mathbf{x}_n\} = 0 + \int_{\mathbf{x}_a}^{\text{basis-states}} d^3x_a\ \hat{N}(\mathbf{x}_a) + \iint_{\{\mathbf{x}_a,\mathbf{x}_b\}} d^3x_a\ d^3x_b\ \hat{N}\{\mathbf{x}_a,\mathbf{x}_b\} + \iiint_{\{\mathbf{x}_a,\mathbf{x}_b,\mathbf{x}_c\}} d^3x_a\ d^3x_b\ d^3x_c\ \hat{N}\{\mathbf{x}_a,\mathbf{x}_b,\mathbf{x}_c\} + \ldots$$

$$\textit{where} \quad \hat{N}\{\mathbf{x}_a,\mathbf{x}_b,\ldots\} = \hat{a}^\dagger(\mathbf{x}_a)\hat{a}(\mathbf{x}_a) + \hat{a}^\dagger(\mathbf{x}_b)\hat{a}(\mathbf{x}_b) + \ldots \qquad \text{and so on .}$$

In the momentum basis, simple replace $\mathbf{x} \to \mathbf{p}$. For example, consider some superposition of the vacuum and a two-particle fermion state. The average number of particles, noting that the vacuum contributes no particles, is:

$$|\phi\rangle = c|0\rangle + \iint_{\{\mathbf{x}_a,\mathbf{x}_b\}} d^3x_a\ d^3x_b\ w\{\mathbf{x}_a,\mathbf{x}_b\}\hat{b}^\dagger(\mathbf{x}_b)\hat{b}^\dagger(\mathbf{x}_a)|0\rangle \qquad \Rightarrow$$

$$\hat{N}\{\mathbf{x}_a,\mathbf{x}_b\} = \hat{b}^\dagger(\mathbf{x}_b)\hat{b}(\mathbf{x}_b) + \hat{b}^\dagger(\mathbf{x}_a)\hat{b}(\mathbf{x}_a) = 2$$

$$\langle N\rangle = \langle\phi|\hat{N}|\phi\rangle = 0 + \iint_{\{\mathbf{x}_a,\mathbf{x}_b\}} d^3x_a\ d^3x_b\ 2\left|w\{\mathbf{x}_a,\mathbf{x}_b\}\right|^2 .$$

Again, the average number of particles need not be an integer.

Normalization, Interpretation, and Amplitude of *N* = 1 Multi-particle States

For multiparticle states, the normalization is a bit subtle, and requires understanding the indistinguishability of identical particles. The normalization of a wave-function, and the probability- or particle-density interpretation of a wave-function, are directly connected. From the normalization, we can

derive how the particle-density depends on the wave-function, or alternatively, from the particle-density we can derive the normalization.

We're still limiting ourselves here to all occupation numbers $N = 0$ or 1. **Recall our notation:** $\phi(\ )$ is a quantum mechanical wave-function, $|\phi\rangle$ is the equivalent quantum field state, and $w(\ )$ is the QFT weighting function. n is the number of identical particles in the state (which is different than N, the excitation of a single mode).

We have a single-page summary of multi-particle states on starting on p93.

In Your Dreams

Because we are connecting NRQM to QFT, for now, we take each quantum state and wave-function to be normalized to 1. In NRQM, and in quantum field theory, the 1-particle versions of all seven of the following equations hold. For simplicity, we'd like to have them hold in the multiparticle case, as well:

$$\left.\begin{array}{ll} \text{(a)} & \int_\infty d^3x_1\, d^3x_2\, |\phi(\mathbf{x}_1, \mathbf{x}_2)|^2 = 1 \\ \text{(b)} & \langle \mathbf{x}_1\mathbf{x}_2 \,|\, \mathbf{x}_1\mathbf{x}_2 \rangle = 1 \\ \text{(c)} & \langle \phi \,|\, \phi \rangle = 1 \end{array}\right\} \quad \text{we declare.} \tag{5.14}$$

We define the above to be so. In analogy with the 1-particle case, we'd also like to have:

$$\left.\begin{array}{ll} \text{(d)} & w(\mathbf{x}_1, \mathbf{x}_2) = \langle \mathbf{x}_1, \mathbf{x}_2 \,|\, \phi \rangle \\ \text{(e, wrong)} & \phi(\mathbf{x}_1, \mathbf{x}_2) = \langle \mathbf{x}_1, \mathbf{x}_2 \,|\, \phi \rangle \\ \text{(f, wrong)} & |\phi(\mathbf{x}_a, \mathbf{x}_b)|^2 = \Pr(\mathbf{x}_a \text{ and } \mathbf{x}_b) \\ \text{(g, wrong)} & \phi(\mathbf{x}_1, \mathbf{x}_2) = w(\mathbf{x}_1, \mathbf{x}_2) \end{array}\right\} \begin{array}{l} \text{(d) we want, and get.} \\ \text{(e), (f), (g) we want, but can't have.} \end{array} \tag{5.15}$$

NB: You can't always get what you want. We show below that the correct equations are:

$$\left.\begin{array}{ll} \text{(e)} & \phi(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{\sqrt{2!}} \langle \mathbf{x}_1, \mathbf{x}_2 \,|\, \phi \rangle \\ \text{(f)} & \phi(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{\sqrt{2!}} w(\mathbf{x}_1, \mathbf{x}_2) \\ \text{(g)} & \Pr(\mathbf{x}_a \text{ and } \mathbf{x}_b) = (2!)|\phi(\mathbf{x}_a, \mathbf{x}_b)|^2 \end{array}\right\} \quad \text{best we can do.}$$

The above equations generalize easily to n-particles: replace (2!) with (n!).

Beware: Almost everyone agrees with the choice (5.15)(b), but not [Bay 19-39 p419]. Also, much of QFT (especially scattering) uses a different, Lorentz-covariant normalization, which we cover in detail elsewhere, as needed.

We now derive the consequences implied by (5.14), and derive the correct equations (d), (e), (f), and (g).

Multi-particle Wave-function Normalization and Density

For two identical particles, one must consider the *joint*-probability-density, since the particles are entangled in the wave-function $\phi(\mathbf{x}_1, \mathbf{x}_2)$. In other words, it is only meaningful to ask: if I measured the locations of the two particles, what is the probability (density) to find one particle at $\mathbf{x}_1$ and another particle at $\mathbf{x}_2$? It is *not* meaningful to ask where I might find "particle 1" and "particle 2," because there is no first or second particle; there are only two indistinguishable particles. That is, it is only meaningful to ask, "What is the probability density of the *configuration* $\{\mathbf{x}_1, \mathbf{x}_2\}$?" From fundamental quantum axioms, this density must be proportional to $|w(\mathbf{x}_1, \mathbf{x}_2)|^2$:

$$\Pr(\text{particles at } \mathbf{x}_1 \text{ and } \mathbf{x}_2) \propto \left|w(\mathbf{x}_1, \mathbf{x}_2)\right|^2 .$$

(5.14)(a) is mathematically convenient, as in NRQM:

$$\int_\infty d^3x_1\, d^3x_2\, \left|\phi(\mathbf{x}_1, \mathbf{x}_2)\right|^2 = 1 ,$$

so we *choose* it as our normalization. This simply says the sum of all probabilities must be 1. (All sources we've seen use this normalization.) This mandates how we compute probability/particle-density. In the integral, both positions $\mathbf{x}_1$ and $\mathbf{x}_2$ cover all space, so each particle-pair (configuration), say $\{\mathbf{x}_a, \mathbf{x}_b\}$, appears twice: once as $\phi(\mathbf{x}_a, \mathbf{x}_b)$ and once as $\phi(\mathbf{x}_b, \mathbf{x}_a)$. For both fermions and boson, anti/symmetry requires those two ϕ values have the same magnitude, and describe the same physical state, so the joint probability density of the configuration $\{\mathbf{x}_a, \mathbf{x}_b\}$ is:

$$\Pr\{\mathbf{x}_a, \mathbf{x}_b\} = \left|\phi(\mathbf{x}_a, \mathbf{x}_b)\right|^2 + \left|\phi(\mathbf{x}_b, \mathbf{x}_a)\right|^2 = 2\left|\phi(\mathbf{x}_a, \mathbf{x}_b)\right|^2 . \tag{5.16}$$

This is the correct (5.15)(f). For n particles, we have $\Pr\{\mathbf{x}_1, \ldots \mathbf{x}_n\} = n!\left|\phi(\mathbf{x}_a, \mathbf{x}_b)\right|^2$.

We can now determine the wave-function $\phi(\mathbf{x}_a, \mathbf{x}_b)$: we have seen that the quantum field weight function $w(\,)$ is distributed over 2 values of the wave-function ϕ (one value of ϕ for each ordering of the configuration $\{\mathbf{x}_1, \mathbf{x}_2\}$). Using our normalizations for $w(\,)$ and $\phi(\,)$, we have:

$$1 = \frac{1}{2}\int_\infty d^3x_1 \int_\infty d^3x_2\ \left|w(\mathbf{x}_1, \mathbf{x}_2)\right|^2 = \int_\infty d^3x_1\, d^3x_2\, \left|\phi(\mathbf{x}_1, \mathbf{x}_2)\right|^2 \quad \Rightarrow$$

$$\left|\phi(\mathbf{x}_1, \mathbf{x}_2)\right| = \frac{1}{\sqrt{2!}}\left|w(\mathbf{x}_1, \mathbf{x}_2)\right|, \tag{5.17}$$

$$\phi(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{\sqrt{2!}}\, w(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{\sqrt{2!}}\left\langle \mathbf{x}_1, \mathbf{x}_2 \middle| \phi \right\rangle .$$

For n particles, $(2!) \rightarrow (n!)$.

Multi-particle Quantum Field State Normalization

(5.14)(b) says we define our basis kets (and bras) normalized to 1. For both bosons and fermions, our definition (5.7) of the ket $|\mathbf{x}_a, \mathbf{x}_b\rangle$, which also defines the bra (5.12), implies immediately that it is normalized to one: $\left\langle \mathbf{x}_a, \mathbf{x}_b \middle| \mathbf{x}_a, \mathbf{x}_b \right\rangle = 1$. (Again, only [Bay 19-39 p419] uses a different convention.)

Since (5.14)(b) is the squared-magnitude of a valid quantum field state, then to be consistent, all QFT states must also be normalized to 1, i.e. (5.14)(c): $\langle\phi|\phi\rangle = 1$. Given our definition (5.13) of a continuous field state, we must define its normalization as:

$$\begin{aligned}\langle\phi|\phi\rangle &\equiv \iint\limits_{\substack{\text{unique} \\ \text{pairs } \mathbf{x}_a, \mathbf{x}_b}} d^3x_a\, d^3x_b\, \underbrace{\langle 0|\hat{b}(\mathbf{x}_a)\hat{b}(\mathbf{x}_b)}_{\langle \mathbf{x}_a, \mathbf{x}_b|}\, w^*(\mathbf{x}_a, \mathbf{x}_b)\, w(\mathbf{x}_a, \mathbf{x}_b)\, \underbrace{\hat{b}^\dagger(\mathbf{x}_b)\hat{b}^\dagger(\mathbf{x}_a)|0\rangle}_{|\mathbf{x}_a, \mathbf{x}_b\rangle} \\ &= \iint\limits_{\substack{\text{unique} \\ \text{pairs } \mathbf{x}_a, \mathbf{x}_b}} d^3x_a\, d^3x_b\, \left|w(\mathbf{x}_a, \mathbf{x}_b)\right|^2 = 1 .\end{aligned}$$

Therefore, the probability density for a particle at $\mathbf{x}_a$ and $\mathbf{x}_b$, from the quantum field state, is:

$$\Pr(\mathbf{x}_a \text{ and } \mathbf{x}_b) = \left|w(\mathbf{x}_a, \mathbf{x}_b)\right|^2 .$$

Note the difference between this and (5.16); the probability formula for the quantum field state differs from probability formula for the QM wave-function by the factor $(n!)$.

With these normalizations and probabilities nailed down, we easily show that (5.15)(d) holds:

$$\langle \mathbf{x}_1, \mathbf{x}_2 | \phi \rangle = \iint\limits_{\substack{\text{unique} \\ \text{pairs } \mathbf{x}_a, \mathbf{x}_b}} d^3x_a \; d^3x_b \; \underbrace{\langle 0 | \hat{b}(\mathbf{x}_1)\hat{b}(\mathbf{x}_2)}_{\langle \mathbf{x}_1, \mathbf{x}_2 |} \; w(\mathbf{x}_a, \mathbf{x}_b) \; \underbrace{\hat{b}^\dagger(\mathbf{x}_b)\hat{b}^\dagger(\mathbf{x}_a)|0\rangle}_{|\mathbf{x}_a, \mathbf{x}_b\rangle}$$

$$= \iint\limits_{\substack{\text{unique} \\ \text{pairs } \mathbf{x}_a, \mathbf{x}_b}} d^3x_a \; d^3x_b \; \delta^3(\mathbf{x}_1 - \mathbf{x}_a)\delta^3(\mathbf{x}_2 - \mathbf{x}_b) w(\mathbf{x}_a, \mathbf{x}_b) = w(\mathbf{x}_1, \mathbf{x}_2) \,.$$

In practice, integrating over unique pairs is mathematically difficult. We can make it simpler by noting that if we integrated *both* $\mathbf{x}_a$ and $\mathbf{x}_b$ over all space, we would count each configuration (unique pair) twice. But such an integral is mathematically more tractable, so *equivalent to* the definition of inner product for field states implied above, we could instead say:

$$\langle \phi | \phi \rangle \equiv \iint\limits_{\substack{\text{unique} \\ \text{pairs } \mathbf{x}_a, \mathbf{x}_b}} d^3x_a \; d^3x_b \; \underbrace{\langle 0 | \hat{b}(\mathbf{x}_a)\hat{b}(\mathbf{x}_b)}_{\langle \mathbf{x}_a, \mathbf{x}_b |} \; w^*\{\mathbf{x}_a, \mathbf{x}_b\} \; w\{\mathbf{x}_a, \mathbf{x}_b\} \; \underbrace{\hat{b}^\dagger(\mathbf{x}_b)\hat{b}^\dagger(\mathbf{x}_a)|0\rangle}_{|\mathbf{x}_a, \mathbf{x}_b\rangle}$$

$$= \frac{1}{2}\int_\infty d^3x_a \int_\infty d^3x_b \; \left| w\{\mathbf{x}_a, \mathbf{x}_b\} \right|^2 = 1 \,.$$

The ½ compensates for counting each configuration twice. For n particles, ½ → $(1/n!)$. (This same method is used to make the Dyson-Wicke perturbation expansion for the scattering matrix.)

The continuous 2-particle field state (5.13) has the same difficulty of integrating over just configurations (unique pairs), so we can apply the same method:

$$|\phi\rangle = \frac{1}{\sqrt{2}}\int_\infty d^3x_a \int_\infty d^3x_b \; w(\mathbf{x}_a, \mathbf{x}_b)\hat{b}^\dagger(\mathbf{x}_b)\hat{b}^\dagger(\mathbf{x}_a)|0\rangle \,,$$

cf. [SBH 13.79 p133]. It is much easier to simply overcount, and then divide by the overcounting factor.

We can also relate the field state $|\phi\rangle$ to the wave-function $\phi(\mathbf{x}_1, \mathbf{x}_2)$, recalling that the probability of any configuration is already split across the wave-function at *both* orderings of $\{\mathbf{x}_a, \mathbf{x}_b\}$:

$$|\phi\rangle = \int_\infty d^3x_a \int_\infty d^3x_b \; \phi(\mathbf{x}_a, \mathbf{x}_b)\hat{b}^\dagger(\mathbf{x}_b)\hat{b}^\dagger(\mathbf{x}_a)|0\rangle \,,$$

This also follows immediately from (5.17): $\phi(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{\sqrt{2!}} w(\mathbf{x}_1, \mathbf{x}_2) \,.$

n-particle States

The generalization from 2 to an n-particle state is straightforward: all the factors of 2! in the 2-particle derivation become $n!$. We now tersely repeat the above derivations, adapted for n particles.

Our quantum field basis states for n particles are:

$$|\mathbf{x}_1, \mathbf{x}_2, ...\rangle \equiv ... \hat{b}^\dagger(\mathbf{x}\;)\hat{b}^\dagger(\mathbf{x}_1)|0\rangle \qquad \Rightarrow \qquad \langle \mathbf{x}_1, \mathbf{x}_2, ... | \mathbf{x}_1, \mathbf{x}_2, ... \rangle = 1$$

We choose our quantum field state $|\phi\rangle$ normalization to be integrated only over n-particle *configurations*:

$$|\phi\rangle = \iiint\limits_{\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, ...\}} d^3x_1 \; d^3x_2 \; d^3x_3 \; ... \; w\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, ...\} \; \underbrace{... \hat{b}^\dagger(\mathbf{x}_3)\hat{b}^\dagger(\mathbf{x}_2)\hat{b}^\dagger(\mathbf{x}_1)|0\rangle}_{|\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, ...\rangle} \qquad \Rightarrow$$

$$\langle \phi | \phi \rangle = \iiint\limits_{\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, ...\}} d^3x_1 \; d^3x_2 \; d^3x_3 \; ... \left| w\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, ...\} \right|^2 = 1, \qquad \text{and} \tag{5.18}$$

$$w\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, ...\} = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, ... | \phi \rangle$$

We can also write $|\phi\rangle$ as an integral over all space for all coordinates by correcting for overcounting:

$$|\phi\rangle = \frac{1}{\sqrt{n!}} \int_\infty d^3x_1 \int_\infty d^3x_2 \int_\infty d^3x_3 \ldots w\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots\} \ldots \hat{b}^\dagger(\mathbf{x}_3)\hat{b}^\dagger(\mathbf{x}_2)\hat{b}^\dagger(\mathbf{x}_1)|0\rangle$$

When a wave-function exists (all $N = 1$), it overcounts each configuration $n!$ times; therefore:

$$\phi(\mathbf{x}_1, \mathbf{x}_2, \ldots) = \frac{1}{\sqrt{n!}} w\{\mathbf{x}_1, \mathbf{x}_2, \ldots\} = \frac{1}{\sqrt{n!}} \langle \mathbf{x}_1, \mathbf{x}_2, \ldots | \phi \rangle \qquad (\text{all } N = 1),$$

because we already have $\phi(\mathbf{x}_1, \mathbf{x}_2) \propto w\{\mathbf{x}_1, \mathbf{x}_2\}$, and ϕ is normalized over all position space for *all* variables $\mathbf{x}_j$. In other words, there are $n!$ ways to permute the $\mathbf{x}$'s in $\phi(\ )$, so the probability density of measuring one particle at each of a set of n points $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots\}$ is:

$$\Pr(\text{particles at } \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots) = \left|w\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots\}\right|^2 = n!\left|\phi(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots)\right|^2 .$$

This implies that the wave-function $\phi(\ )$ is normalized over all space for all coordinates, as:

$$\int_\infty d^3x_1 \int_\infty d^3x_2 \int_\infty d^3x_3 \ldots \left|\phi(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots)\right|^2 = 1$$

Momentum basis: Here again, for the momentum basis, simply replace $\mathbf{x} \rightarrow \mathbf{p}$ in all of the above.

Another Angle on Field States

We can write the field state $|\phi\rangle$ as a corrected sum of a wave-function ϕ over all the permutations of the same configuration. Here, $P \equiv$ sign of the permutation, the plus sign is for bosons, the minus for fermions:

$$|\phi\rangle = \frac{1}{\sqrt{n!}} \left[\sum_{\substack{\text{permutations} \\ \mathbf{x}_1, \mathbf{x}_2, \ldots}} (\pm 1)^P \phi(\mathbf{x}_1, \mathbf{x}_2, \ldots) |\mathbf{x}_1, \mathbf{x}_2, \ldots\rangle \right].$$

For fermions, any symmetric component in $\phi(\ )$ cancels from the alternating signs in $(-1)^P$, so a fermion field defined this way depends only on the anti-symmetric part of the function $\phi(\ )$ [Bay p421]. Similarly for bosons, any *anti*-symmetric component in $\phi(\ )$ cancels from the above summation of permutations. Thus a boson field defined as such depends only on the *symmetric* part of the function $\phi(\ )$.

Of course, if $\phi(\ )$ had any component of the wrong symmetry, we shouldn't call it a "wave-function".

This completes the discussion of single-excitation multiparticle states. This is sufficient for most scattering work.

When First We Practice ... Symmetry and Entanglement

superposition of tensor products??

n identical particles are *necessarily* entangled by anti/symmetry (it is impossible to *not* be entangled).

The anti/symmetry (fermion/boson) of the wave-functions demands that the particles affect each other, even if there is no interaction term in the hamiltonian. For example, the Pauli exclusion principle, in the more precise form of the anti-symmetry requirement on the wave-function, means that fermions "repel" each other. Statistical Mechanics courses are replete with exercises concerning "non-interacting" fermions. What they mean is "non-energetically-interacting" fermions, i.e. no interaction in the hamiltonian. But identical particles *always* "interact" in the sense that anti/symmetry requirements demand entanglement, which imposes constraints on the allowed configurations of particles, which has observable consequences.

Discrete Basis Systems

A brief aside: some systems have discrete basis states, such as EM cavities where the basis states are the classical resonant modes of the cavity. Then the field state and normalization are sums rather than integrals, in the usual way. For example, a 2-particle field state with discrete bases $\mathbf{p}_j$ satisfies:

$$|\phi\rangle = \sum_{\{a,b\}}^{\text{configurations}} w_{ab}\, \hat{a}^\dagger(\mathbf{p}_b)\hat{a}^\dagger(\mathbf{p}_a)|0\rangle, \qquad \textit{with} \qquad \sum_{\{a,b\}}^{\text{configurations}} |w_{ab}|^2 = 1;$$

$w_{ab} \equiv$ complex amplitudes.

Many references use discrete bases heavily. Continuous bases are often derived as the continuum limit of discrete bases.

Bosons, Multiple Excitations, and Occupation Numbers

Until now, we have considered only states in which all the excitations are $N = 1$. In the **x** basis, this means a multi-particle state $|\mathbf{x}_a, \mathbf{x}_b, \ldots\rangle$ in which all the **x** are distinct. However, for some *boson* systems, multiple excitations of a single mode (basis state) are a realistic possibility:

$$|1_\mathbf{x}\rangle, |2_\mathbf{x}\rangle, |3_\mathbf{x}\rangle, \ldots \qquad (\text{boson excitations of a single basis state}).$$

For example, cavities in a conductor are discrete-basis systems with resonant photon modes that are easily excited to $N > 1$. A continuous-mode example is a laser, which (ideally) produces an EM coherent state, which is a superposition of all excitations of a single mode [Michelsen 2014 p291]. (A coherent state is the closest quantum analog to a classical EM wave.)

> Multiple excitations of a basis state do not exist in NRQM.
> Such field states are describable only in quantum field theory.

This section does not apply to fermions, whose occupation numbers are always either 0 or 1.

Wave-functions are not defined for any $N \geq 2$, but there are still *weight* functions $w\{\,\}$.

Multiple excitations of a single mode require different handling than single excitations; our above simple definition of a multi-particle state (5.7) does not have the required prefactor for multiple excitations as in (5.3). A normalized N-particle state of a single basis mode **x** is:

$$|N\rangle \equiv \frac{1}{\sqrt{N!}}\left(\hat{a}^\dagger_\mathbf{x}\right)^N |0\rangle, \quad N = 0,1,2,\ldots, \quad \langle N|N\rangle = 1. \quad \text{E.g.,} \quad |3_\mathbf{x}\rangle = \frac{1}{\sqrt{3!}}\left(\hat{a}^\dagger_\mathbf{x}\right)^3 |0\rangle.$$

The factorial prefactor isn't really normalization; it's just adjusting for how the raising operator is defined in (5.2) with a factor of $\sqrt{N}$. This prefactor is *completely different* than the similar-looking normalization factor in (5.17); NB: N is the excitation level of a single mode, whereas n is the total number of particles in the field state.

For example, the following 3-boson field state has 2 bosons localized at $\mathbf{x}_a$, and one at $\mathbf{x}_b$. It is normalized, because it is created directly from raising operators:

$$\left|2_{\mathbf{x}_a}1_{\mathbf{x}_b}\right\rangle = \frac{\left(\hat{a}^\dagger(\mathbf{x}_a)\right)^2}{\sqrt{2!}}\hat{a}^\dagger(\mathbf{x}_b)|0\rangle \qquad \text{and} \qquad \left\langle 2_{\mathbf{x}_a}1_{\mathbf{x}_b}\middle|2_{\mathbf{x}_a}1_{\mathbf{x}_b}\right\rangle = 1. \tag{5.19}$$

We define M as the number of modes that are excited; in this example, $M = 2$: the modes $\mathbf{x}_a$ and $\mathbf{x}_b$. We write this field state not as a 3-particle version of (5.13), but as (5.19): 2 modes of excitation. We see that for field states with any $N > 1$, along with the raising operators, we must divide out the factorials from the occupation numbers of each mode $N_1, N_2, N_3, \ldots$. In the above example, there are two excited modes, so we have two occupation numbers: $N_1 = 2$, $N_2 = 1$, and $n = 3 = N_1 + N_2$. For each occupation number N_j, we must remove the factor of $(N_j!)^{1/2}$ that the raising operators introduce.

Generalizing: for an n-particle set of M excited modes, with occupation numbers $\{N_1, N_2, \ldots N_M\}$, we have a normalized boson field state in the **x** basis:

$$|\phi\rangle = \frac{\left[\hat{a}^\dagger(\mathbf{x}_n)\right]^{N_M} \ldots \left[\hat{a}^\dagger(\mathbf{x}_2)\right]^{N_2}\left[\hat{a}^\dagger(\mathbf{x}_1)\right]^{N_1}}{\sqrt{N_1!\,N_2!\,\ldots N_M!}}|0\rangle \qquad [\text{GAF 4.122 p321}]. \tag{5.20}$$

A more general state is a weighted superpositions of these kinds of states. The state-space spanned by these more-general states is called **Fock space**.

For all $N_j = 1$, then the number of particles $n = M$, and this equation reduces to our prior field state form, e.g. (5.7).

In the momentum basis, we simply replace $\mathbf{x} \rightarrow \mathbf{p}$.

Writing Field States in Unnatural Ways: Change of Basis

As in NRQM, when choosing a basis to write a field state, we often pick one of special significance to the system at hand. For an EM cavity, we often choose the cavity modes as a "natural" basis for writing the field state. These basis states are standing waves (Figure 5.3a), each of which is a superposition of two momenta, **p** and –**p**. In that basis, we can readily perform many useful calculations. But what if we want to compute the probability of detecting the photon at a specific detector at position **x**? Then we need the amplitude for the E-field at the point in space **x**. In other words, we need the amplitude $\langle \mathbf{x}|\phi\rangle$ for the component of $|\mathbf{x}\rangle$ in the EM field state $|\phi\rangle$, i.e. the amplitude for $|\mathbf{x}\rangle$ in the position basis. This is a change of basis from **p** to **x**. We discuss here how to find such an amplitude.

Figure 5.3 (a) A photon in a cavity; detectors distributed across the cavity measure the location of the photon interaction. (b) Similarly, spaced detectors localize a traveling photon's interaction.

[Aside: Note that we can localize a photon's interaction to a region of space *arbitrarily smaller* than its wavelength. This is a common point of confusion. In a sense, we cannot localize a single photon to a region smaller than its wavelength, but we can localize its interaction (with say, an atom) to a region *much* smaller than a wavelength.]

Single-particle Basis Change

For simplicity, rather than cavity modes, let's consider just momentum eigenstates. How would we write a boson field state $|\mathbf{p}_a\rangle$ in the **x** basis? $|\mathbf{p}_a\rangle$ is the field state of 1 boson of definite momentum $\mathbf{p}_a$. With $\hbar = 1$, and knowing the wave-function for a momentum eigenstate is $\exp(i\mathbf{p}\bullet\mathbf{x})$, we can use (5.6), $\phi(\mathbf{x}) = w(\mathbf{x})$. Ignoring normalization:

$$\left|1_{\mathbf{p}_a}\right\rangle = \int_\infty d^3x\, e^{i\mathbf{p}_a\bullet\mathbf{x}} \left|1_{\mathbf{x}}\right\rangle \quad \Rightarrow \qquad \left\langle \mathbf{x}\middle|\mathbf{p}_a\right\rangle = e^{i\mathbf{p}_a\bullet\mathbf{x}}. \tag{5.21}$$

From this, we infer the momentum raising operator, written in the **x**-basis (for either bosons or fermions):

$$\left|1_{\mathbf{p}}\right\rangle = \hat{a}^\dagger(\mathbf{p})|0\rangle = \int_\infty d^3x\, e^{i\mathbf{p}\bullet\mathbf{x}} \hat{a}^\dagger(\mathbf{x})|0\rangle \qquad \Rightarrow \qquad \hat{a}^\dagger(\mathbf{p}) = \int_\infty d^3x\, e^{i\mathbf{p}\bullet\mathbf{x}} \hat{a}^\dagger(\mathbf{x}).$$

Multi-particle Basis Change

It is useful to recognize that multiple identical particle field states are similar to tensor products of the single-mode field states:

$$\left|1_{\mathbf{x}_a} 1_{\mathbf{x}_b}\right\rangle \sim \left|1_{\mathbf{x}_b}\right\rangle \left|1_{\mathbf{x}_a}\right\rangle \qquad \text{in the position basis (for fermions, note the ordering);}$$

$$\left|1_{\mathbf{p}_a} 1_{\mathbf{p}_b}\right\rangle \sim \left|1_{\mathbf{p}_b}\right\rangle \left|1_{\mathbf{p}_a}\right\rangle \qquad \text{in the momentum basis.}$$

Thence, a pseudo "product state" of two identical particles, say $|b\rangle|a\rangle$, means that the first mode is in state $|a\rangle$ *and* the second mode is in state $|b\rangle$. It is a combined state encompassing both $|a\rangle$ *and* $|b\rangle$. The combination of two weighted kets is weighted by the product of the "factor" weights:

$$w_b|b\rangle w_a|a\rangle = w_b w_a |b\rangle|a\rangle = w_b w_a |a,b\rangle.$$

Fermions: What is the quantum field state for two fermions $|\mathbf{p}_a\ \mathbf{p}_b>$, written in the **x**-basis? We can always go back to the raising operator(s) acting on the vacuum:

$$\left|\mathbf{p}_a\ \mathbf{p}_b\right\rangle = \hat{b}^\dagger(\mathbf{p}_b)\hat{b}^\dagger(\mathbf{p}_a)\left|0\right\rangle \overset{\text{almost}}{=} \underbrace{\int_\infty d^3x_b\, e^{i\mathbf{p}_b\bullet\mathbf{x}_b}\hat{b}^\dagger(\mathbf{x}_b)}_{\hat{b}^\dagger(\mathbf{p}_b)\text{ in }\mathbf{x}\text{-basis}} \underbrace{\int_\infty d^3x_a\, e^{i\mathbf{p}_a\bullet\mathbf{x}}\hat{b}^\dagger(\mathbf{x})}_{\hat{b}^\dagger(\mathbf{p}_a)\text{ in }\mathbf{x}\text{-basis}}\left|0\right\rangle .$$

The "almost" is because we shouldn't integrate *both* coordinates over all space; as we did when normalizing, eq. (5.18), we should integrate only over (unique) configurations:

$$\left|\mathbf{p}_a\ \mathbf{p}_b\right\rangle = \hat{b}^\dagger(\mathbf{p}_b)\hat{b}^\dagger(\mathbf{p}_a)\left|0\right\rangle = \iint\limits_{\{\mathbf{x}_a,\mathbf{x}_b\}} \underbrace{d^3x_b\, e^{i\mathbf{p}_b\bullet\mathbf{x}_b}\hat{b}^\dagger(\mathbf{x}_b)}_{\hat{b}^\dagger(\mathbf{p}_b)\text{ in }\mathbf{x}\text{-basis}} \underbrace{d^3x_a\, e^{i\mathbf{p}_a\bullet\mathbf{x}_a}\hat{b}^\dagger(\mathbf{x}_a)}_{\hat{b}^\dagger(\mathbf{p}_a)\text{ in }\mathbf{x}\text{-basis}}\left|0\right\rangle \qquad \text{(rearrange:)}$$

$$= \iint\limits_{\{\mathbf{x}_a,\mathbf{x}_b\}} d^3x_b\, d^3x_a\, e^{i\mathbf{p}_b\bullet\mathbf{x}_b} e^{i\mathbf{p}_a\bullet\mathbf{x}_a}\hat{b}^\dagger(\mathbf{x}_b)\hat{b}^\dagger(\mathbf{x}_a)\left|0\right\rangle \qquad (5.22)$$

$$= \iint\limits_{\{\mathbf{x}_a,\mathbf{x}_b\}} d^3x_b\, d^3x_a\, \underbrace{e^{i(\mathbf{p}_b\bullet\mathbf{x}_b+\mathbf{p}_a\bullet\mathbf{x}_a)}}_{w\{\mathbf{x}_a,\mathbf{x}_b\}}\left|\mathbf{x}_a\mathbf{x}_b\right\rangle$$

This implies that $w\{\mathbf{x}_a,\mathbf{x}_b\} = e^{i(\mathbf{p}_b\bullet\mathbf{x}_b+\mathbf{p}_a\bullet\mathbf{x}_a)}$ (unnormalized). The particle number is two, (almost) everywhere and in all bases.

There is a subtlety that is not very important: where $\mathbf{x}_a = \mathbf{x}_b$, the two fermion raising operators give $\mathbf{0}_\mathbf{v}$. However, for infinite-dimensional systems (the most common case), such configurations contribute nothing to an integral over configurations, and so can be ignored.

Bosons: How do we write the field state of 2 boson excitations in the single mode $\mathbf{p}_a$, in the **x** basis? It's the same as (5.21), but with 2 excitations at each point. Essentially, we just use the raising operator on the vacuum twice. Though we aren't normalizing here, remember that the raising operator introduces a factor of √2 that, in principle, we must remove. There is only one excited mode ($M = 1$), so we have only one position coordinate, **x**:

$$\left|2_{\mathbf{p}_a}\right\rangle = \frac{1}{\sqrt{2}}\hat{a}^\dagger(\mathbf{p}_a)\left|1_{\mathbf{p}_a}\right\rangle = \frac{1}{\sqrt{2}}\underbrace{\int_\infty d^3x\, e^{i\mathbf{p}_a\bullet\mathbf{x}}\hat{a}^\dagger(\mathbf{x})}_{\hat{a}^\dagger(\mathbf{p})\text{ in }\mathbf{x}\text{-basis}}\underbrace{e^{i\mathbf{p}_a\bullet\mathbf{x}}\left|1_\mathbf{x}\right\rangle}_{\left|1_{\mathbf{p}_a}\right\rangle} = \int_\infty d^3x\, e^{i2\mathbf{p}_a\bullet\mathbf{x}}\left|2_\mathbf{x}\right\rangle .$$

$$\textit{where} \quad \left|2_\mathbf{x}\right\rangle = \frac{1}{\sqrt{2}}\hat{a}^\dagger(\mathbf{x})\left|1_\mathbf{x}\right\rangle$$

This field state is an eigenstate of the total number operator (in both **p** and **x** bases), with eigenvalue 2. This means the number of particles, everywhere and in every basis, is exactly 2.

The covariant form of this state, $e^{-i\,2\,p_\mu x^\mu}$, includes the time-energy dependence. It shows that, compared to a single excitation, two particles double both the energy and momentum. Equivalently, the temporal frequency $\omega \equiv 2p_a^0$ is doubled, as is the spatial wave vector $\mathbf{k} = 2\mathbf{p}_a$. Lorentz covariance demands that time and space go together. You might be troubled that $|N_\mathbf{p} = 2>$ seems to have half the wavelength (and twice the frequency) of $|N_\mathbf{p} = 1>$, even though all the excitations of the same mode have the same physical (observable) wavelength (consider photons, e.g.). However, the electric field operator is such that the observable wavelength and frequency are always the classical ones [GAF 5.53 p356], regardless of the weight-function wavelength. This raises an important point:

> The "wavelength" of the quantum weight function can be different than the measurable wavelength of the particle.

This same issue arises in NRQM, where the quantum wave-function is gauge-dependent, and so its wavelength may differ from the observable wavelength.

How do we write the field state of 2 bosons in $\mathbf{p}_a$ and 1 boson in $\mathbf{p}_b$, in the $\mathbf{x}$ basis? It's the "product" of the two field states (still unnormalized). We have two excited modes ($M = 2$), so we integrate over unique pairs $\{\mathbf{x}_a, \mathbf{x}_b\}$:

$$\begin{aligned}\left|2_{\mathbf{p}_a} 1_{\mathbf{p}_b}\right\rangle &\propto \iint_{\{\mathbf{x}_a,\mathbf{x}_b\}} d^3x_a\, e^{i\,2\mathbf{p}_a\cdot\mathbf{x}_a}\left|2_{\mathbf{x}_a}\right\rangle d^3x_b\, e^{i\mathbf{p}_b\cdot\mathbf{x}_b}\left|1_{\mathbf{x}_b}\right\rangle \\ &= \iint_{\{\mathbf{x}_a,\mathbf{x}_b\}} d^3x_a\, d^3x_b\, e^{i\left(2\mathbf{p}_a\cdot\mathbf{x}_a+\mathbf{p}_b\cdot\mathbf{x}_b\right)}\left|2_{\mathbf{x}_a}\right\rangle\left|1_{\mathbf{x}_b}\right\rangle\end{aligned}$$

This state is an eigenstate of the total number operator (in any basis), with eigenvalue 3: the number of particles everywhere is exactly 3.

As in the fermion case (5.22), there is the subtle issue of when $\mathbf{x}_a = \mathbf{x}_b$: then the state is $|3_{\mathbf{x}a}>$, and the raising operator introduces a factor of $\sqrt{3}$. Again, this is a set of measure 0, so it can be ignored.

The General Quantum Field State

A proper understanding of QFT requires understanding the most general states of a quantum field. This is a little tedious, and in practice, we usually use simpler states for computation. However, the full generality of quantized fields is needed for problems such as QFT-perturbed states of atoms, because they have an uncertain number of photons.

In a previous section, we defined quantum field states with a definite number of particles. The most general quantum field state is a superposition of component states, each with a different number of particles. Such a field state has an uncertain number of particles.

For simplicity, and direct connection to NRQM, this section uses weight-function normalization, as described earlier: the norm of the quantum field state $<\phi|\phi> = 1$. Beware, though, that much of QFT uses a different, Lorentz-invariant normalization, which we cover in detail elsewhere, as needed.

The General Identical Fermion Field State

We start with identical fermions, because their states are simpler than bosons. We again start with the position basis, because we find it conceptually the easiest. The most general field state is simply a superposition of the states of all possible definite particle numbers: 0, 1, 2, ... A general quantum field state is an infinite superposition of a 0-particle state (the vacuum), a 1-particle state (5.6), a 2-particle state (5.13), etc. without bound. This is somewhat clumsy to write. At a given time t:

$$\begin{aligned}\left|\Psi\right\rangle = a\left|0\right\rangle &+ \int_{\infty} d^3x\, w_1(\mathbf{x})\left|1_{\mathbf{x}}\right\rangle \\ &+ \iint_{\{\mathbf{x}_1,\mathbf{x}_2\}} d^3x_1\, d^3x_2\, w_2(\mathbf{x}_1,\mathbf{x}_2)\left|1_{\mathbf{x}_1}1_{\mathbf{x}_2}\right\rangle \\ &+ \iiint_{\{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\}} d^3x_1\, d^3x_2\, d^3x_3\, w_3(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)\left|1_{\mathbf{x}_1}1_{\mathbf{x}_2}1_{\mathbf{x}_3}\right\rangle \\ &+ \ldots\end{aligned} \quad . \tag{5.23}$$

[cf SBH 13.79 p133]. Such a field state cannot be written as a wave-function of any kind, and does not have a numeric value at a point $\mathbf{x}$. Instead, it has a quantum state at every point $\mathbf{x}$, because each point $\mathbf{x}$ is a quantum oscillator.

The probability of measuring exactly a given number of particles in the field state $|\Psi>$ is:

$$\Pr\big(\text{0-particles (vacuum)}\big) = |a|^2$$

$$\Pr\big(\text{1-particle}\big) = \int_\infty d^3x\, |w_1(\mathbf{x})|^2$$

$$\Pr\big(\text{2-particles}\big) = \iint_{\{\mathbf{x}_1,\mathbf{x}_2\}} d^3x_1\, d^3x_2\, |w_2(\mathbf{x}_1,\mathbf{x}_2)|^2$$

$$\Pr\big(\text{3-particles}\big) = \iiint_{\{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\}} d^3x_1\, d^3x_2\, d^3x_3\, |w_3(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)|^2,$$

and so on.

In the case of a single-particle NR system, the field state has exactly 1 particle, and its wave-function is $\phi_1(\mathbf{x}) = w_1(\mathbf{x})$, which must be normalized as in NRQM. All the other $w_n\{...\}$ are zero. We see again that the single-particle wave-function can be obtained from the quantum field state by taking the inner product:

$$\phi(\mathbf{x}) = w_1(\mathbf{x}) = \langle 0|\hat{b}(\mathbf{x})|\Psi\rangle = \langle 1_{\mathbf{x}}|\Psi\rangle \qquad \big(\text{single-particle field state}\big).$$

(The last equality is from $\hat{b}$ acting to the left.) If the field has any components beyond w_1 (including the vacuum), then it has an uncertain number of particles, and cannot be represented as a wave-function. In terms of a (toy) field operator, say $\hat{\psi}(\mathbf{x}) = \hat{b}^\dagger(\mathbf{x}) + \hat{b}(\mathbf{x})$, the above equation can be written:

$$\phi(\mathbf{x}) = w_1(\mathbf{x}) = \langle 0|\hat{\psi}(\mathbf{x})|\Psi\rangle = \langle 1_x|\Psi\rangle \qquad \big(\text{single-particle field state}\big).$$

Note that only the $\hat{b}$ term in $\hat{\psi}$ contributes.

In the NRQM case of two identical particles, the field state has exactly 2 particles, and their joint wave-function is given by (5.17): $\phi(\mathbf{x}_1, \mathbf{x}_2) = w_2(\mathbf{x}_1, \mathbf{x}_2)/\sqrt{2}$, which will already be normalized (if Ψ is). All the other $w_n\{...\}$ are zero. As in the single-particle case, the joint wave-function can be found from the quantum field operator $\hat{\psi}(\mathbf{x}) = \hat{b}^\dagger(\mathbf{x}) + \hat{b}(\mathbf{x})$, because only the $\hat{b}$ terms in $\hat{\psi}$ contribute:

$$\phi(\mathbf{x}_1,\mathbf{x}_2) = \frac{1}{\sqrt{2!}}\, w_2(\mathbf{x}_1,\mathbf{x}_2) = \frac{1}{\sqrt{2!}}\langle 0|\hat{\psi}(\mathbf{x}_2)\hat{\psi}(\mathbf{x}_1)|\Psi\rangle = \frac{1}{\sqrt{2!}}\langle 0|\hat{b}(\mathbf{x}_2)\hat{b}(\mathbf{x}_1)|\Psi\rangle.$$

Higher-*n* multi-particle wave-functions are straightforward generalizations of the two-particle case.

General identical-fermion field state normalization: Recall that the $w\{...\}$ are normalized without any factorials, as in (5.18). At a time *t*, the general state $\Psi(\mathbf{x})$ must be normalized such that the probabilities for each number of particles *sum to* 1:

$$\Pr(\text{vacuum}) + \Pr(\text{1-particle}) + \Pr(\text{2-particles}) + \ldots = 1,$$

or [SBH 13-85 p135]:

$$1 = |\Psi|^2 = |a|^2 + \int_\infty d^3x\, |w_1(\mathbf{x})|^2 + \iint_{\{\mathbf{x}_1,\mathbf{x}_2\}} d^3x_1\, d^3x_2\, |w_2(\mathbf{x}_1,\mathbf{x}_2)|^2 + \iiint_{\{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\}} d^3x_1\, d^3x_2\, d^3x_3\, |w_3(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)|^2 + \ldots .$$

Momentum basis: In scattering, we usually work in the momentum basis. General identical-fermion field states of momentum are just like the position basis, but with $\mathbf{x} \to \mathbf{p}$:

$$|\Psi\rangle = a|0\rangle + \int_\infty d^3p\, w_1(\mathbf{p})|1_\mathbf{p}\rangle \qquad \left(where\ |1_\mathbf{p}\rangle \equiv \hat{b}^\dagger(\mathbf{p})|0\rangle\right)$$
$$+\iint_{\{\mathbf{p}_1,\mathbf{p}_2\}} d^3p_1\, d^3p_2\, w_2(\mathbf{p}_1,\mathbf{p}_2)|1_{\mathbf{p}_1}1_{\mathbf{p}_2}\rangle$$
$$+\iiint_{\{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\}} d^3p_1\, d^3p_2\, d^3p_3\, w_3(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)|1_{\mathbf{p}_1}1_{\mathbf{p}_2}1_{\mathbf{p}_3}\rangle$$
$$+\dots\ .$$

The General Identical Boson Field State

General bosons field states are conceptually similar to their fermion counterparts, but have the added tedium of an infinity of multiple excitations of each basis mode. Recall the boson single-particle notation of (5.3):

$$|1_\mathbf{x}\rangle \equiv \hat{a}^\dagger(\mathbf{x})|0\rangle, \quad |2_\mathbf{x}\rangle \equiv \frac{\left(\hat{a}^\dagger(\mathbf{x})\right)^2}{\sqrt{2!}}|0\rangle, \qquad \dots\ |N_\mathbf{x}\rangle \equiv \frac{\left(\hat{a}^\dagger(\mathbf{x})\right)^N}{\sqrt{N!}}|0\rangle\ .$$

Eq. (5.20) displays a boson n-particle field state for a given set of occupation numbers $\{N_j\}$. To compose the general boson field state from (5.20), we extend our notation for fermions, (5.23), to allow a sum over all sets of occupation numbers $\{N_{\mathbf{x}j}\}$:

$$|\Psi\rangle = a|0\rangle + \int_\infty d^3x\, w_1(\mathbf{x})|1_\mathbf{x}\rangle$$
$$+\sum_{\{N_{\mathbf{x}_1},N_{\mathbf{x}_2}\}} \iint_\infty d^3x_1\, d^3x_2\, \frac{w\left(\{N_{\mathbf{x}_1},N_{\mathbf{x}_2}\};\mathbf{x}_1,\mathbf{x}_2\right)}{\sqrt{N_{\mathbf{x}_1}!\,N_{\mathbf{x}_2}!}}\left|N_{\mathbf{x}_1}N_{\mathbf{x}_2}\right\rangle$$
$$+\sum_{\{N_1,N_2,N_3\}} \iiint_\infty d^3x_1\, d^3x_2\, d^3x_3\, \frac{w\left(\{N_1,N_2,N_3\};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\right)}{\sqrt{N_1!\,N_2!\,N_3!}}\left|N_{\mathbf{x}_1}N_{\mathbf{x}_2}N_{\mathbf{x}_3}\right\rangle$$
$$+\dots$$
$$+\sum_{\{N_j\}} \iiint_\infty d^3x_1\, d^3x_2\, d^3x_3 \dots \frac{w\left(\{N_j\};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\dots\right)}{\sqrt{N_1!\,N_2!\,N_3!\dots N_M!}}\left|N_{\mathbf{x}_1}N_{\mathbf{x}_2}N_{\mathbf{x}_3}\dots\right\rangle \quad +\dots.$$

where $M \equiv$ # of excited modes. The summations are over all sets $\{N_j\}$ with $N_j \geq 1$ and $\Sigma N_j = n$. With bosons, the $\mathbf{x}$'s are allowed to be duplicated, which goes along with $N_j > 1$.

A simple example of a (very important) state that is a superposition of all excitations is the quasi-classical state, aka "coherent state" [Michelsen 2014 "Quasi-classical States"]. This state is the ideal "laser state": the closest you can get to an ideal pure sinusoidal wave. A coherent state is defined by a single complex number α, and the classical mode ($\boldsymbol{\varepsilon}$, $\mathbf{k}$):

$$|\alpha\rangle \equiv \underbrace{\exp\left(-|\alpha|^2/2\right)}_{\text{normalization}}\left[|0\rangle + \alpha|1\rangle + \frac{\alpha^2}{\sqrt{2!}}|2\rangle + \dots \frac{\alpha^N}{\sqrt{N!}}|N\rangle + \dots\right] \qquad \text{(quasi-classical state)}$$
$$= \exp\left(-|\alpha|^2/2\right)\sum_{N=0}^{\infty}\frac{\alpha^N}{\sqrt{N!}}|N\rangle$$

Normalization of a general boson field state is conceptually the same as for fermions: all the component weight function norms must sum to 1:

$$1 = |\Psi|^2 = |a|^2 + \int_\infty d^3x\, |w_1(\mathbf{x})|^2$$

$$+ \sum_{\{N_1,N_2\}} \int_{\{\mathbf{x}_1,\mathbf{x}_2\}} d^3x\, \left|w\left(\{N_1,N_2\};\mathbf{x}_1,\mathbf{x}_2\right)\right|^2$$

$$+ \sum_{\{N_1,N_2,N_3\}} \iiint_{\{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\}} d^3x_1\, d^3x_2\, d^3x_3\, \left|w\left(\{N_1,N_2,N_3\};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\right)\right|^2 + \ldots$$

$$+ \ldots \quad + \underbrace{\sum_{\{N_j\}} \int_{\{\mathbf{x}_1,\mathbf{x}_2,\ldots\mathbf{x}_M\}} d^3x_1\, d^3x_2\, d^3x_3 \ldots \left|w\left(\{N_j\};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\ldots\right)\right|^2}_{n\text{-particle}} + \ldots.$$

Recall that when integrating over (unique) configurations $\{\mathbf{x}_1, \ldots \mathbf{x}_M\}$, the $w\{\ldots\}$ are normalized *without* any factorials, as in (5.18).

Momentum basis: In the momentum basis, a "mode" is a momentum **p**. All the above carries over into the momentum basis by simple taking $\mathbf{x} \to \mathbf{p}$.

The Quantum Field State of the Universe

So far, we've considered only the state of a single field at a time: a photon field, or an electron field, etc. Of course, the state the of universe is a combination of the states of *all* the fields that exist. Starting small, the input or output state of most scattering experiments involves particles of different types. For example, the input to electron-neutrino ($e^-\ \nu_e$) scattering is a tensor product state, usually of definite momenta and particle number:

$$|in\rangle = \underbrace{|\mathbf{p}_a\rangle}_{e^-}\ \underbrace{|\mathbf{p}_b\rangle}_{\nu_e} = \underbrace{\hat{b}^\dagger_{e^-,\mathbf{p}_a}|0\rangle}_{e^-}\ \underbrace{\hat{b}^\dagger_{\nu_e,\mathbf{p}_b}|0\rangle}_{\nu_e} \qquad \text{scattering input state .}$$

This is similar to the two components of (5.1), which are each a tensor product of the electron state and the photon state. The order of the raising operators doesn't matter because the e^- and ν are not identical, so the $\hat{b}^\dagger$ commute.

We show in later chapters that a particle and its anti-particle together compose a single field. For example, the electron and positron together compose the "electron field."

Of course, a state of multiple particle types (which we might call a "multi-type state") can be a tensor product of any combination of the physical fields: 6 leptons (e, μ, τ, ν_e, ν_μ, ν_τ), photon, W, Z, 18 quarks (6 flavors $\times$ 3 colors), and 8 gluons (we ignore the Higgs). Such tensor products are the basis states of the universe. They are too complicated to write down precisely, and there is little need to do so. However, we can build on the field states of this chapter to show a *basis* state of the universe in a simplified notation:

$$\left|\psi_{universe,basis}\right\rangle = w\left(\left\{N_{j,type}\right\}\mathbf{x}_1,\mathbf{x}_2,\ldots\right)\left|N_{\text{basis-1,type-1}}\right\rangle\left|N_{\text{basis-2,type-2}}\right\rangle\cdots\left|N_{\text{basis-m,type-m}}\right\rangle,$$

where each $|N_{\text{basis-j,type-j}}\rangle$ is an occupation number of a basis state of a particle type, and particle types may be repeated (for different occupation numbers).

The complete state of the universe is the (entangled) superposition of such basis states:

$$\left|\Psi_{universe}\right\rangle = \sum_{\substack{\text{all tensor}\\ \text{products}\\ \text{of fields}\\ j,type}} w\left(\left\{N_{j,type}\right\}\mathbf{x}_1,\mathbf{x}_2,\ldots\right)\left|N_{\text{basis-1,type-1}}\right\rangle\left|N_{\text{basis-2,type-2}}\right\rangle\cdots\left|N_{\text{basis-m,type-m}}\right\rangle.$$

Thus the quantum field state of the universe is rather complicated.

