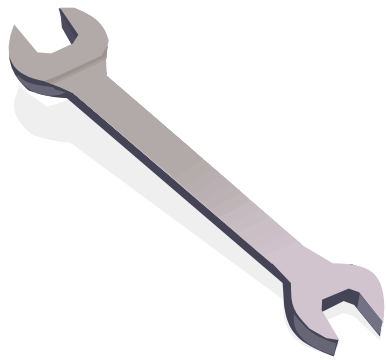
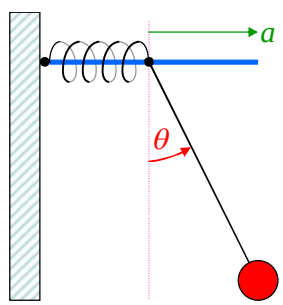
Funky Mechanics Concepts

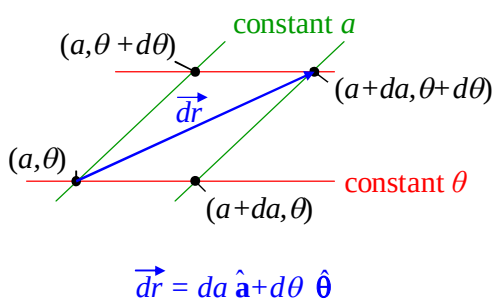


The Anti-Textbook*

A work in progress. See physics.ucsd.edu/~emichels for the latest versions of the Funky Series.
Please send me comments.

Eric L. Michelsen





"Now in the further development of science, we want more than just a formula. First we have an observation, then we have numbers that we measure, then we have a law which summarizes all the numbers. But the real *glory* of science is that *we can find a way of thinking* such that the law is *evident*."

– Richard Feynman, *The Feynman Lectures on Physics*, p26-3.

* Physical, conceptual, geometric, and pictorial physics that didn't fit in your textbook.

Please do NOT distribute this document. Instead, link to physics.ucsd.edu/~emichels/FunkyMechanicsConcepts.pdf.

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2006 values from NIST. For more physical constants, see http://physics.nist.gov/cuu/Constants/ .

Speed of light in vacuum	c = 299 792 458 m s^{-1} (exact)
Gravitational constant	G = 6.674 28(67) x 10^{-11} m^3 kg^{-1} s^{-2}
Relative standard uncertainty	±1.0 x 10^{-4}
Boltzmann constant	k = 1.380 6504(24) x 10^{-23} J K^{-1}
Stefan-Boltzmann constant	σ = 5.670 400(40) x 10^{-8} W m^{-2} K^{-4}
Relative standard uncertainty	±7.0 x 10^{-6}
Avogadro constant	N_A, L = 6.022 141 79(30) x 10^{23} mol^{-1}
Relative standard uncertainty	±5.0 x 10^{-8}
Molar gas constant	R = 8.314 472(15) J mol^{-1} K^{-1}
calorie	4.184 J (exact)
Electron mass	m_e = 9.109 382 15(45) x 10^{-31} kg
Proton mass	m_p = 1.672 621 637(83) x 10^{-27} kg
Proton/electron mass ratio	m_p/m_e = 1836.152 672 47(80)
Elementary charge	e = 1.602 176 487(40) x 10^{-19} C
Electron g-factor	g_e = –2.002 319 304 3622(15)
Proton g-factor	g_p = 5.585 694 713(46)
Neutron g-factor	g_N = –3.826 085 45(90)
Muon mass	m_μ = 1.883 531 30(11) x 10^{-28} kg
Inverse fine structure constant	α^{-1} = 137.035 999 679(94)
Planck constant	h = 6.626 068 96(33) x 10^{-34} J s
Planck constant over 2π	$\hbar$ = 1.054 571 628(53) x 10^{-34} J s
Bohr radius	a_0 = 0.529 177 208 59(36) x 10^{-10} m
Bohr magneton	μ_B = 927.400 915(23) x 10^{-26} J T^{-1}

Other values:

1 inch ≡ 0.0254 m (exact)

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1 Introduction

Why Funky?

The purpose of the "Funky" series of documents is to help develop an accurate physical, conceptual, geometric, and pictorial understanding of important physics topics. We focus on areas that don't seem to be covered well in most texts. The Funky series attempts to clarify those neglected concepts, and others that seem likely to be challenging and unexpected (funky?). The Funky documents are intended for serious students of physics; they are not "popularizations" or oversimplifications.

Physics includes math, and we're not shy about it, but we also don't hide behind it.

Without a conceptual understanding, math is gibberish.

This work is one of several aimed at graduate and advanced-undergraduate physics students. Go to http://physics.ucsd.edu/~emichels for the latest versions of the Funky Series, and for contact information. We're looking for feedback, so please let us know what you think.

How to Use This Document

This work is not a text book.

There are plenty of those, and they cover most of the topics quite well. This work is meant to be used *with* a standard text, to help emphasize those things that are most confusing for new students. When standard presentations don't make sense, come here.

You should read all of this introduction to familiarize yourself with the notation and contents. After that, this work is meant to be read in the order that most suits you. Each section stands largely alone, though the sections are ordered logically. Simpler material generally appears before more advanced topics. You may read it from beginning to end, or skip around to whatever topic is most interesting. The "Shorts" chapter is a diverse set of very short topics, meant for quick reading.

If you don't understand something, read it again *once*, then keep reading.
Don't get stuck on one thing. Often, the following discussion will clarify things.

The index is not yet developed, so go to the web page on the front cover, and text-search in this document.

Notation

See the glossary for a list of common terms.

Notations used throughout the Funky Series:

Important points are highlighted in blue boxes.

Tips to help remember or work a problem are sometimes given in green boxes.

Common misconceptions are sometimes written in dark red dashed-line boxes.

TBS stands for "To Be Supplied," i.e., I'm working on it. Let me know if you want it now.

?? For this work in progress, double question marks indicates areas that I hope to further expand in the final work. Reviewers: please comment especially on these areas, and others that may need more expansion.

[Square brackets] in text indicate asides: interesting points that can be skipped without loss of continuity. They are included to help make connections with other areas of physics.

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[Asides may also be shown in smaller font and narrowed margins. Notes to myself may also be included as asides.]

Formulas: When we list a function's argument as q_i, we mean there are n arguments, q_1 through q_n.

We write the integral over the entire domain as a subscript "∞", for any number of dimensions:

1-D: $\int_\infty dx$ 3-D: $\int_\infty d^3x$

Evaluation between limits: we use the notation $[function]_a^b$ to denote the evaluation of the function between a and b, i.e.,

$[f(x)]_a^b \equiv f(b) - f(a)$. For example, $\int_0^1 3x^2\, dx = [x^3]_0^1 = 1^3 - 0^3 = 1$

We write the probability of an event as "Pr(event)."

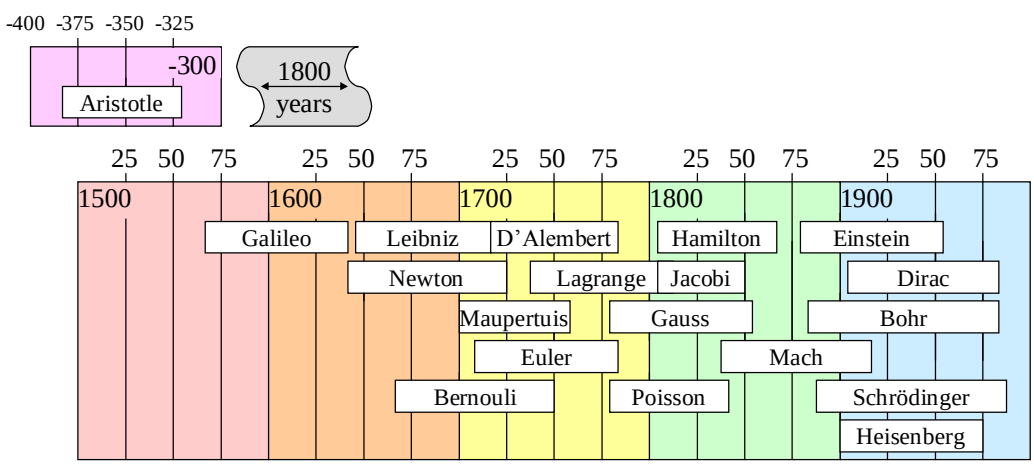
In my word processor, I can't easily make fractions for derivatives, so I sometimes use the standard notation d/dx and $\partial/\partial x$.

Vector variables: In some cases, to emphasize that a variable is a vector, it is written in bold; e.g., $V(\mathbf{r})$ is a scalar function of the vector, $\mathbf{r}$. $\mathbf{E}(\mathbf{r})$ is a vector function of the vector, $\mathbf{r}$.

Column vectors: Since it takes a lot of room to write column vectors, but it is often important to distinguish between column and row vectors. I sometimes save vertical space by using the fact that a column vector is the transpose of a row vector:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = (a,b,c,d)^T$$

Scientists of the Times



Timeline of important scientists.

The Funky Series

The purpose of the "Funky" series of documents is to help develop an accurate physical, conceptual, geometric, and pictorial understanding of important physics topics. We focus on areas that don't seem to be covered well in any text we've seen. The Funky documents are intended for serious students of physics. They are not "popularizations" or oversimplifications, though they try to start simply, and build to more advanced topics. Physics includes math, and we're not shy about it, but we also don't hide behind it.

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Without a conceptual understanding, math is gibberish.

This work is one of several aimed at graduate and advanced-undergraduate physics students. I have found many topics are consistently neglected in most common texts. This work attempts to fill those gaps. It is *not* a text in itself. You must use some other text for many standard presentations.

The "Funky" documents focus on what is glossed over in most texts. They seek to fill in the gaps. They are intended to be used with your favorite text on the subject. We include many references to existing texts for more information.

Thank You

I owe a big thank you to many professors at both SDSU and UCSD, for their generosity even when I wasn't a real student: Dr. Herbert Shore, Dr. Peter Salamon, Dr. George Fuller, Dr. Andrew Cooksy, Dr. Arlette Baljon, Dr. Tom O'Neil, Dr. Terry Hwa, and others. Thanks also to Yaniv Rosen and Jason Leonard for their many insightful comments and suggestions.

The International System of Units (SI)

The abbreviation SI is from the French: "Le Système international d'unités (SI)", which means "The International System of Units." This is the basis for all modern science and engineering. To understand some of these constants, you must be familiar with the basic physics involving them. The SI system defines seven units of measure, mostly using repeatable methods reproducible in a sophisticated laboratory. The exception is the kilogram, which requires a single universal standard kilogram prototype be preserved; it is in France.

[From http://physics.nist.gov/cuu/Units/current.html, with my added notes.] The following table of definitions of the 7 SI base units is taken from NIST Special Publication 330 (SP 330), *The International System of Units (SI).*

Unit of length	meter[3]	The meter is the length of the path travelled [*sic*] by light in vacuum during a time interval of 1/299 792 458 of a second.
Unit of mass	kilogram	The kilogram is the unit of mass; it is equal to the mass of the international prototype of the kilogram.
Unit of time	second[1]	The second is the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium 133 atom.
Unit of electric current	ampere[2]	The ampere is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 meter apart in vacuum, would produce between these conductors a force equal to 2×10^{-7} newton per meter of length.
Unit of thermodynamic temperature	kelvin	The kelvin, unit of thermodynamic temperature, is the fraction 1/273.16 of the thermodynamic temperature of the triple point of water.
Unit of amount of substance	mole	1. The mole is the amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon 12; its symbol is "mol." 2. When the mole is used, the elementary entities must be specified and may be atoms, molecules, ions, electrons, other particles, or specified groups of such particles.

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Unit of luminous intensity	candela	The candela is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540 x 10^{12} hertz and that has a radiant intensity in that direction of 1/683 watt per steradian.

My added notes:

1. As of January, 2002, NIST's latest primary cesium standard was capable of keeping time to about 30 nanoseconds per year ($1 \cdot 10^{-15}$).
2. The effect of the definition of "ampere" is to fix the magnetic constant (permeability of vacuum) at exactly $\mu_0 = 4\pi \times 10^{-7}$ H · m^{-1}.
3. This *defines* the speed of light as exactly 299 792 458 m/s.

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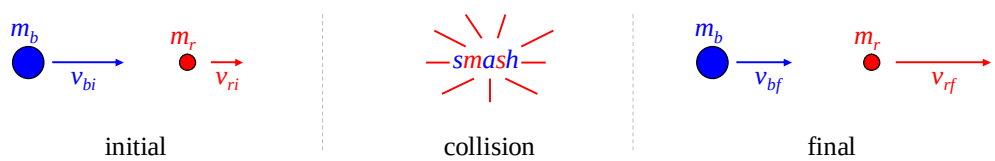
2 Symmetries, Coordinates

Elastic Collisions

This section illustrates how understanding physical principles simplifies a problem, and provides broadly applicable results. We follow these steps:

1. Pose a problem of general interest
2. Transform it to a simpler problem, using a physical principle
3. Solve the simpler problem
4. Generalize the result to all nonrelativistic frames
5. Generalize to 2D collisions in the COM frame
6. Generalize to relativistic collisions

1. A problem of general interest: Consider a 1-dimensional (1D) **elastic** collision, i.e. one in which *kinetic* energy is conserved. Recall that *all* processes conserve total energy, but *elastic* processes conserve *kinetic* energy. Let a heavy blue mass collide with a lighter red mass. Suppose they both move to the right, but the blue moves faster. Eventually it collides with the red. It might look like this:



Elastic collision where the blue particle collides with the red particle from behind.

What can we say about the initial and final velocities? From physical symmetries, we can quickly find an important result: the closure rate before the collision equals separation rate after the collision. To see this, we use the symmetry:

Physics is the same for all observers.

We define an observer's **frame of reference** as the observer's state of motion and the coordinate system fixed to her. So an "observer" is the same thing as a "frame of reference." All observers have the same laws of physics. In fancy talk, we could say "Physics is *invariant* over all frames of reference." There is a bit of a fuss about *inertial* frames vs. non-inertial frames, and we recognize that:

Physics is often simpler in inertial frames of reference.

2. Transform to a simpler problem: Now let's use the symmetry of invariance over frames of reference to simplify the above collision, and determine the final velocities. It's often simplest to eliminate as much motion as possible, so we choose the **center of mass frame** (COM), i.e. the observer moves at the velocity of the COM. Therefore, to the observer, the center of mass does not move. Since total momentum equals (total mass) times (COM velocity), total momentum in the COM frame is always zero:

$$\mathbf{p}_{total} = M_{total}\mathbf{v}_{COM} = \left(M_{total}\right)(\mathbf{0}_{\mathbf{v}}) = 0 \qquad \textit{where} \quad \mathbf{0}_{\mathbf{v}} \equiv (0,0,0) = \text{the zero vector}\ .$$

In the COM frame, our collision looks like this:

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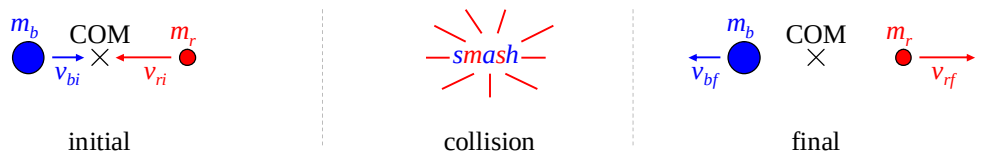


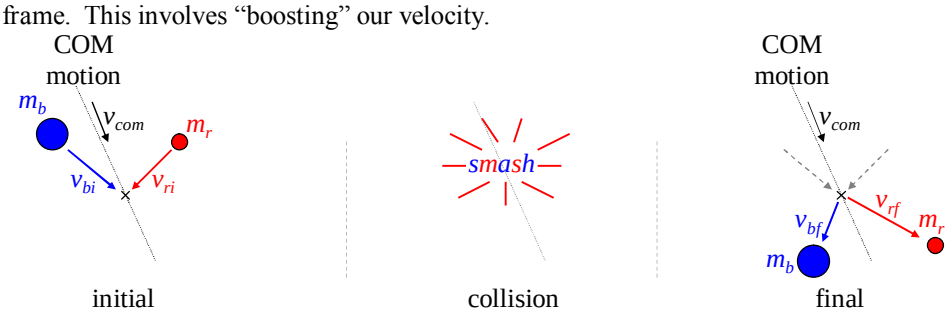
Elastic collision viewed in COM frame.

The red speed in the COM frame is higher than the blue speed, because red and blue have the same magnitude of momentum, but in opposite directions.

3. Solve the simpler problem: We see immediately that for total momentum to be zero, blue and red must move in opposite directions, both before and after the collision. Furthermore, the magnitudes of their momenta must be equal. This is our first constraint on the final velocities. [Note that we don't need to know it, but nonrelativistically, each particle's speed must be inversely proportional to its mass.] Our second constraint is to conserve kinetic energy (elastic collision). So we have two unknowns (final velocities), and two constraints, therefore, there is a unique solution. The initial velocities satisfy these constraints: that's how we identified the constraints. Since kinetic energy depends only on the magnitude of velocity, but not its sign, reversing the two velocities preserves the two particles' total momentum (zero), and their kinetic energies, and therefore satisfies both constraints. It is the unique answer (derived with no algebra).

We notice that in the COM frame, the closure speed (before the collision) equals the separation speed (after the collision).

4. Generalize the result: As observer, we now move to an arbitrary inertial frame, not the COM frame. This involves "boosting" our velocity.



Elastic collision viewed in a boosted frame.

In the new frame, both blue and red particles have different velocities. The new velocities are the COM velocities plus some constant [the negative of our boost velocity]. But the *difference* between the blue and red velocities is the same in *any* frame: the rate of closure is the same for *all* observers. E.g., if we timed how long it takes for them to collide, we get the same time in any (nonrelativistic) frame. Similarly, the rate of separation is the same for *all* observers. So we see:

Using a simple physical principle, we arrive at a universal result.

To wit: for an elastic collision, the rate of closure before the collision equals the rate of separation after the collision, for all observers. Notice that the collision is 1D in the COM frame, but is 2D in our boosted frame.

Of course, using algebra, we can get the 1D result in any frame collinear with the COM motion by starting with conservation of momentum, and conservation of kinetic energy:

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$$m_b v_{bi} + m_r v_{ri} = m_b v_{bf} + m_r v_{rf} \quad \Rightarrow \quad m_b\left(v_{bi} - v_{bf}\right) = m_r\left(v_{rf} - v_{ri}\right)$$

$$\frac{1}{2}m_b v_{bi}^2 + \frac{1}{2}m_r v_{ri}^2 = \frac{1}{2}m_b v_{bf}^2 + \frac{1}{2}m_r v_{rf}^2 \quad \Rightarrow \quad m_b\left(v_{bi}^2 - v_{bf}^2\right) = m_r\left(v_{rf}^2 - v_{ri}^2\right)$$

$$\cancel{m_b\left(v_{bi} - v_{bf}\right)}\left(v_{bi} + v_{bf}\right) = \cancel{m_r\left(v_{rf} - v_{ri}\right)}\left(v_{rf} + v_{ri}\right)$$

$$v_{bi} + v_{bf} = v_{rf} + v_{ri}$$

$$v_{bi} - v_{ri} = v_{rf} - v_{bf}$$

I find this obtuse, and not very enlightening. We can make it more insightful by labeling the math:

$$m_b v_{bi} + m_r v_{ri} = m_b v_{bf} + m_r v_{rf} \quad \Rightarrow \quad \underbrace{m_b\left(v_{bi} - v_{bf}\right)}_{\text{momentum lost by blue}} = \underbrace{m_r\left(v_{rf} - v_{ri}\right)}_{\text{momentum gained by red}}$$

$$\frac{1}{2}m_b v_{bi}^2 + \frac{1}{2}m_r v_{ri}^2 = \frac{1}{2}m_b v_{bf}^2 + \frac{1}{2}m_r v_{rf}^2 \quad \Rightarrow \quad \underbrace{m_b\left(v_{bi}^2 - v_{bf}^2\right)}_{\text{energy lost by blue}} = \underbrace{m_r\left(v_{rf}^2 - v_{ri}^2\right)}_{\text{energy gained by red}}$$

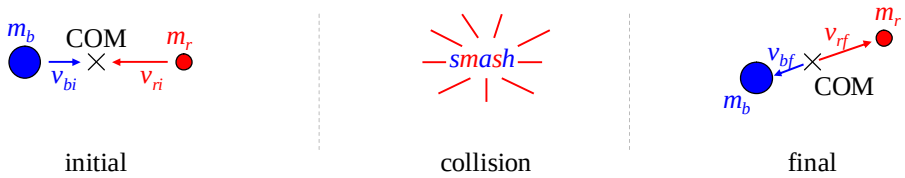
Factoring: $\cancel{m_b\left(v_{bi} - v_{bf}\right)}\left(v_{bi} + v_{bf}\right) = \cancel{m_r\left(v_{rf} - v_{ri}\right)}\left(v_{rf} + v_{ri}\right)$

Canceling: $v_{bi} + v_{bf} = v_{rf} + v_{ri}$

Rearranging: $\underbrace{v_{bi} - v_{ri}}_{\text{initial closure rate}} = \underbrace{v_{rf} - v_{bf}}_{\text{final separation rate}}$

Better, but this is still nowhere near as simple or as general as the result using physical and mathematical principles, but no algebra.

5. Generalize to 2D collisions in the COM frame: The nice thing about the physical symmetry approach (physics is the same in any frame) is that it allows us to generalize the 1D result with virtually no effort. Consider a collision which is already 2D in the COM frame, where the particles glance off each other, and bounce off at an angle:



Two-dimensional Elastic collision viewed in COM frame.

The same argument still applies: there exists a COM frame, the relative velocities are still frame independent. The two momenta must be opposite at all times, and the kinetic energy is a function only of the magnitudes of the velocities. Therefore the separation rate equals the closure rate. The algebraic approach also works for 2D and 3D (for a boosted observer): in fact, the algebra is nearly identical, except we replace the numbers v_b and v_r with *vectors* $\mathbf{v}_b$ and $\mathbf{v}_r$.

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6. Generalize to relativistic mechanics: Now here's the *big* payoff: what about a relativistic collision, where the particles move at nearly the speed of light? The diagram is qualitatively the same as above, but the algebra is much harder, what with $\gamma(v) \equiv 1/\sqrt{1-v^2/c^2}$, and all. Most importantly, the symmetries are similar:

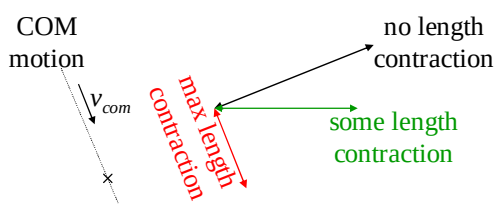
(1) Physics is the same for all observers.

(2) Momentum is parallel to velocity, and kinetic energy depends only on the magnitude of velocity. Therefore, in the COM frame, reversing the velocities reverses the momenta, and leaves kinetic energies invariant.

We can see these symmetries from the mathematical forms for relativistic momentum and kinetic energy:

$$\mathbf{p} = \gamma(v) m\mathbf{v}, \qquad E = \left(\gamma(v) - 1\right) mc^2 \qquad \textit{where} \quad v \equiv |\mathbf{v}|, \quad \textit{and} \quad \gamma(v) \equiv \frac{1}{\sqrt{1-v^2/c^2}}$$

We must consider more carefully, though, the effect of boosting to a new observer frame, because of time-dilation and length-contraction. Time dilation is independent of direction of motion, but length-contraction depends on direction:



Length contraction relative to COM frame, for different directions.

We consider 3 possible cases of observer moving w.r.t the COM:

(a) For a small boost, where the observer moves slowly with respect to the COM frame, we can neglect time-dilation and length-contraction. Therefore, the result still holds, even if the particles move relativistically with respect to each other.

(b) For a relativistic observer w.r.t the COM, but one collinear with both particle motions, the collision appears 1D to the observer (the particles hit head on and retrace their paths after the collision). However, the magnitudes of the velocities change after the collision, so our earlier argument doesn't hold. Instead, we note that the time dilation and length contraction factors (between observer and COM frames) are the same before and after the collision. Therefore, the closure rate and separation rate are both different from the COM values, but different in the same way. In other words, if rate of closure = rate of separation in the COM, then the closure distance per unit time is still the same as the separation distance per unit time in the observer frame.

(c) Not so with 2D collisions (2D as seen by the observer): the relativistic observer sees the separating particles go off at a different angle than they approached. In this case, length contraction relative to the COM frame is different in different directions. Hence, the separation gets a different length contraction than the approach, but time is dilated the same for both. Therefore, the separation speed is *different* than the closure speed.

Recapping: for relativistic particles, and for relativistic observers of 1D collisions, we get the same general result as nonrelativistically: the velocity of closure before the collision equals the velocity of separation afterward. We also showed, without algebra, that the result does *not* hold for 2D or 3D collisions as viewed by a relativistic observer: in these cases, the rate of closure does *not* equal the rate of separation. If we really want to understand these cases, we can resort to algebra, or some greater set of physical symmetries (of which I'm not aware).

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Summary: Using simple physical and mathematical principles, but no algebra, we established the result that in all cases but one, elastic collisions have a closure rate (before the collision) equal to the separation rate (after the collision). This is true even relativistically, except for the case where the observer is relativistic with respect to the COM frame, and the collision appears 2D to the observer. The use of symmetry saves substantially more algebra in the relativistic case than the nonrelativistic case.

Newton's Laws

Applicable when $v << c$, and $\Delta p\ \Delta x >> \hbar$.

Newton's 3rd Law Isn't

Isn't a law, that is. Newton's 3rd law says that for every force, there is an equal but opposite reaction force. It implies conservation of momentum:

$$dp_1 = F_1\ dt \qquad dp_2 = F_2\ dt \qquad F_2 = -F_1 \qquad \Rightarrow \qquad dp_1 = -dp_2 \qquad and \qquad dp_1 + dp_2 = 0$$

But Newton's 3rd law is wrong: consider magnetic forces that 2 positive particles exert on each other:

B_1 is the magnetic field created by the motion of q_1. Similarly for B_2. (Left) $B_1 = 0$ along the line of motion of q_1, so q_2 feels no force, but q_1 does. (Right) Both particles feel a force, but they are not equal and opposite.

On the right, the two forces are not opposite.

This failure of the 3rd law, and the apparent non-conservation of momentum, are only significant for relativistic speeds, because the Coulomb force between the particles is much greater than the magnetic force. The magnetic force is lower by a factor of v/c. However, conservation of momentum is fully restored when we include the effects of EM radiation [Gri ??]. When the particles accelerate, they radiate. The radiation itself carries momentum, which exactly equals the missing momentum from the accelerated particles. We do not show this result here.

Also note that the third law also does not apply to relativistic systems, because it implies instantaneous force at a distance. Here, again, the physical reality of the EM field is necessary to conserve energy, momentum, and the speed of light as the maximum signal speed.

It's Got Potential

The work energy theorem says that the change in kinetic energy of a particle resulting from a force equals the work done by that force:

$$W \equiv \int_a^b F\ dx = \Delta KE \qquad or \qquad W \equiv \int_a^b \mathbf{F} \bullet d\mathbf{r} = \Delta KE \qquad \text{(higher dimensions)}\ .$$

The work-energy theorem implies that conservative forces (those whose work between two points is independent of the path taken between the points) can be written as the (negative) gradient of a scalar potential:

$$F(\mathbf{r}) = -\nabla U(\mathbf{r}) \qquad \text{(conservative force)}\ ,$$

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where the force is a function of position.

Dissipative forces, friction and drag, are not conservative, and cannot be written with potentials.

On the other hand, workless forces (magnetic and Coriolis forces) always act perpendicular to the velocity, and thus cannot do work [we neglect the work done by magnetic fields on the intrinsic dipole moments of fundamental particles such as electrons]. Workless forces can be written as vector potentials:

$$\mathbf{F} = k\mathbf{v} \times \mathbf{B} = k\mathbf{v} \times \left(\nabla \times \mathbf{A}\right) \qquad \textit{where} \quad \mathbf{A} \equiv \text{vector potential}.$$

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3 Rotating Stuff

Angular Displacements and Angular Velocity Vectors

The following facts underlie the analysis of rotating bodies:

Finite angular displacements are *not* vectors, but angular velocities *are* vectors.

In this case, the time derivative of a non-vector is, in fact, a vector. Here's why:

In 3 or more dimensions, consider rotations about axes fixed in space. Finite angular displacements (i.e., rotations) are *not* vectors because adding (composing) two finite angular displacements is *not* commutative; the result depends on the order in which the rotations are made. In other words, finite angular displacements do not **commute**. (Compare to translational displacements which *are* vectors: moving first a meters in x and then b in y yields the same point as first moving b in y and then a in x.) Therefore, finite angular displacements do *not* compose a vector space because you cannot decompose a finite rotation into "component" rotations relative to some basis rotations.

However, infinitesimal angular displacements *do* commute. Rotating first by $d\theta$ from the z-axis, and then $d\phi$ around the z-axis yields the same rotation as rotating $d\phi$ around the z-axis and then $d\theta$ around the y-axis (i.e., from the z-axis toward the x-axis) (to first order) ??No it doesn't??. Therefore, infinitesimal rotations *do* compose a vector space, and you *can* decompose any infinitesimal rotation into the composition of component rotations relative to some basis rotations. We cannot legitimately associate a vector with a finite rotation. I know, I hear the protests now about "axis of rotation" and finite angles; the point is that there is no *basis* into which we can decompose arbitrary finite rotations. The "sum" (i.e., composition) of two rotations is *not* the vector sum of the two rotation "vectors".

This latter point yields an interesting consequence: whereas finite rotations are *not* vectors, angular velocities *are* vectors. Whereas finite rotations do *not* compose a vector space, angular velocities *do* compose a vector space. This is because angular velocities consist of a large number of infinitesimal rotations occurring in infinitesimal time periods: $d\theta/dt$, and infinitesimal rotations *are* vectors.

Furthermore, these results generalize:

For any set of generalized coordinates, even if the $\{q_i\}$ do not compose a vector, the $\{\dot{q}_i\}$ *do* compose a vector.

Central Forces

Central force problems are widespread in both classical and quantum mechanics. The prototypical central force is either gravity or electrostatics, where the force is a function of the magnitude of the separation of the bodies. A **central force** is a force between 2 bodies which acts along the line of their separation; it may be attractive or repulsive, and its magnitude depends only on the distance between the bodies. Two-body problems are especially common, and their solution is greatly simplified by a technique called "reduction to 1-body". A typical 2-body problem is shown in Figure 3.1, left.

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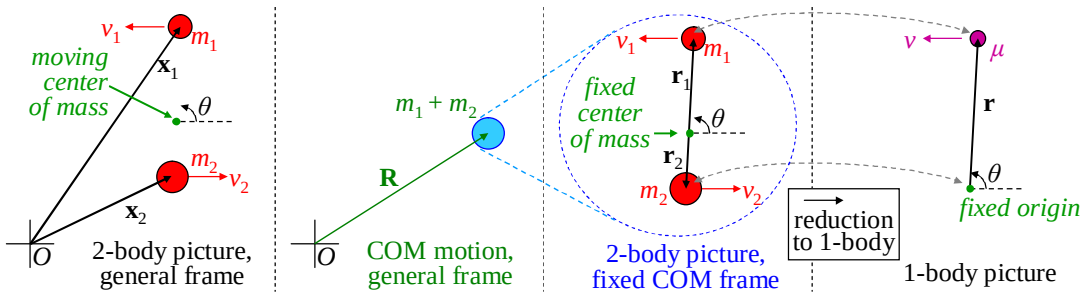

Figure 3.1 (Left) A typical 2-body central force problem. (Right) The equivalent 1-body problem. Reduction to 1-body preserves *mechanics*, but not other physics, such as E&M.

There are typically up to 3 steps in solving central force problems:

1. If we start with a central force 2-body problem, reduce it to a 1-body problem.
2. Replace the 1-body 2D (r, θ) problem with a 1D r problem, by introducing $V_{eff}(r)$ and a fixed angular momentum parameter, l.
3. To solve for orbits, we often introduce a fixed energy parameter, E, and a fixed angular momentum, l.

Reduction To 1-Body

Reduction to 1-body is a change of generalized coordinates. It is *not* just a change of reference frame to the center-of-mass frame, because a translation of reference frame does not change the number of bodies in the problem. A simple change to the CM frame would *still* have two bodies, and they both move. Reduction to 1-body is valid for all formulations of mechanics: Newtonian, Lagrangian, and Hamiltonian. In all cases, the equations of motion separate into non-interacting equations for separate coordinates. Reduction to 1-body can be thought of as a canonical transformation from ($\mathbf{x}_1$, $\mathbf{p}_1$, $\mathbf{x}_2$, $\mathbf{p}_2$) to ($\mathbf{R}$, $\mathbf{P}$, $\mathbf{r}$, $\mathbf{p}$), where ($\mathbf{R}$, $\mathbf{P}$) is CM motion, and ($\mathbf{r}$, $\mathbf{p}$) is relative motion. We often discard ($\mathbf{R}$, $\mathbf{P}$), if it is either motionless, or trivial. Reduction to 1-body also applies to quantum mechanics [Bay 7-11 p171b].

The goal of reduction to 1-body is to construct a simpler, but equivalent, mechanical system to solve a 2-body central force problem. To mimic the 2-body problem, our 1-body reduced problem must reproduce the following properties at all times of the motion:

1. Have the same separation from the origin as the two bodies from each other, to have the same (central) force;
2. Have the same moment of inertia, to have the same angular behavior;
3. Have the same radial acceleration, to have the same radial behavior;
4. Have the same total energy.

It is not obvious that such a transformation is possible. Note that the separation of bodies, $\mathbf{r}(t)$, varies with time. We will find that replicating $\mathbf{r}(t)$, and the moment of inertia at all times reproduces *all* the orbiting motion, and satisfies all the conditions above.

To satisfy condition one, we simply declare it by fiat: the one body problem will have its one body at position $\mathbf{r}(t)$ such that:

$$\mathbf{r}(t) = \mathbf{r}_1(t) - \mathbf{r}_2(t)\,. \tag{3.1}$$

Because $\mathbf{r}_1$ and $\mathbf{r}_2$ point in opposite directions, in magnitudes we have:

$$r(t) = r_1(t) + r_2(t)\,. \tag{3.2}$$

We must then choose the **reduced mass** so that it produces the same moment of inertia (condition 2), if possible. We first compute the 2-body center of mass, and its total moment of inertia:

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$$m_1 r_1 = m_2 r_2 \qquad \Rightarrow \qquad r_2 = \frac{m_1}{m_2} r_1 \qquad\qquad where \quad r_1 \equiv |\mathbf{r}_1|, \; etc.$$

$$I = m_1 r_1^2 + m_2 r_2^2 = m_1 r_1^2 + m_2 \left(\frac{m_1}{m_2} r_1 \right)^2 = m_1 r_1^2 \left(1 + \frac{m_1}{m_2} \right)$$

We find the reduced mass which produces this moment of inertia:

$$I = \mu r^2 = \mu (r_1 + r_2)^2 = \mu \left(r_1 + \frac{m_1}{m_2} r_1 \right)^2 = \mu r_1^2 \left(1 + \frac{m_1}{m_2} \right)^2 = m_1 r_1^2 \left(1 + \frac{m_1}{m_2} \right).$$

From the last equality, the r_1^2 cancel, and we have:

$$\mu = m_1 \left(1 + \frac{m_1}{m_2} \right)^{-1} = m_1 \left(\frac{m_2 + m_1}{m_2} \right)^{-1} = \frac{m_1 m_2}{m_1 + m_2} \qquad or \qquad \mu = \left(\frac{1}{m_1} + \frac{1}{m_2} \right)^{-1}.$$

This is excellent, because the reduced mass is *independent* of $\mathbf{r}(t)$, and therefore independent of time. At this point, we have no more freedom to choose our 1-body parameters. We have satisfied conditions 1 and 2, and must now verify that conditions 3 are 4 are met.

The 2-body radial acceleration is:

$$a_1 = \frac{F_{12}}{m_1}, \qquad a_2 = \frac{F_{21}}{m_2} = \frac{F_{12}}{m_2}, \quad where \quad F_{12} \equiv \text{magnitude of force on } m_1 \text{ due to } m_2, \text{ etc.}$$

$$a \equiv \ddot{r}(t) = a_1 + a_2 = F_{12} \left(\frac{1}{m_1} + \frac{1}{m_2} \right) \equiv \frac{F_{12}}{m_{effective}} \qquad \Rightarrow$$

$$m_{effective} = \left(\frac{1}{m_1} + \frac{1}{m_2} \right)^{-1} = \frac{m_1 m_2}{m_1 + m_2} = \mu$$

It's a miracle! The very same reduced-mass which satisfies the rotational equation, satisfies the radial equation, as well.

If only the total energy (condition 4) holds up. The potential energy is the same, since it depends only on radial separation, and we *defined* that to be the same for 1-body as 2-body. The kinetic energies are:

$$T_{2-body} = \frac{1}{2} \left(m_1 \dot{r}_1^2 + m_2 \dot{r}_2^2 + I \dot{\theta}^2 \right) = \frac{1}{2} \left(m_1 \dot{r}_1^2 + m_2 \left(\frac{m_1}{m_2} \dot{r}_1 \right)^2 + I \dot{\theta}^2 \right) = \frac{1}{2} \left(m_1 \dot{r}_1^2 \left(1 + \frac{m_1}{m_2} \right) + I \dot{\theta}^2 \right)$$

$$T_{1-body} = \frac{1}{2} \left(\mu \dot{r}^2 + I \dot{\theta}^2 \right). \qquad\qquad \text{The first term is:}$$

$$\mu \dot{r}^2 = \frac{m_1 m_2}{m_1 + m_2} \left(\dot{r}_1 + \dot{r}_2 \right)^2 = \frac{m_1 m_2}{m_1 + m_2} \dot{r}_1^2 \left(1 + \frac{m_1}{m_2} \right)^2 = m_1 \dot{r}_1^2 \frac{m_2}{m_1 + m_2} \left(\frac{m_1 + m_2}{m_2} \right)^2 = m_1 \dot{r}_1^2 \left(1 + \frac{m_1}{m_2} \right)$$

Lo, and behold! The 1-body energy is the same as the 2-body. All the mechanics is preserved.

Note that the one-body equivalent problem is good for the *mechanics* only, but not for, say, electromagnetics. E.g., consider a moving charge near an identical, oppositely moving charge in the center-of-mass frame.

From E&M, we know the system has no dipole radiation, because the dipole moment is constant:

$$\mathbf{d} = e_1 \mathbf{r}_1 + e_2 \mathbf{r}_2 = e \left(\mathbf{r}_1 + \mathbf{r}_2 \right) \qquad where \quad e \equiv \text{electric charge}$$

$$\text{But} \qquad m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 = m \left(\mathbf{r}_1 + \mathbf{r}_2 \right) = const \quad \Rightarrow \quad \left(\mathbf{r}_1 + \mathbf{r}_2 \right) = const \quad \Rightarrow \quad \mathbf{d} = const$$

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However, if we reduce to a 1-body problem, we have a single accelerating charge, but no way to assign a magnitude to that charge. In fact, there is no charge we can use to reproduce the electromagnetics, because the 1-body has a varying dipole moment **d**, which therefore radiates with a dipole component. Hence:

The 1-body reduction is good for *mechanics* only,
not for other physics, such as electromagnetics.

Also, the reduced system does *not* reproduce linear motion of **R** and **P**, as $\mu \neq M \equiv m_1 + m_2$.

To return to the real 2-body motion, we consider Figure 3.1 (right). We can see almost by inspection that:

$$\mathbf{x}_1 = \mathbf{R} + \frac{m_2}{m_1 + m_2}\mathbf{r}, \qquad \text{and} \qquad \mathbf{x}_2 = \mathbf{R} - \frac{m_1}{m_1 + m_2}\mathbf{r},$$

because each body has a distance to the center of mass which is inversely proportional to its mass.

Effective Potential

As soon as we speak of "effective potential, V_{eff}", we assume some *fixed* angular momentum l as a parameter of the problem. (Of course, with no external torques, the angular momentum is indeed fixed.) Then the **effective potential** is just the *real* potential *plus* the kinetic energy due to the fixed angular momentum:

$$V_{eff} \equiv V(r) + \frac{l^2}{2mr^2}, \quad or \quad V(r) + \frac{p_\theta{}^2}{2I}, \qquad where \quad p_\theta \equiv l, \text{ the angular momentum}$$

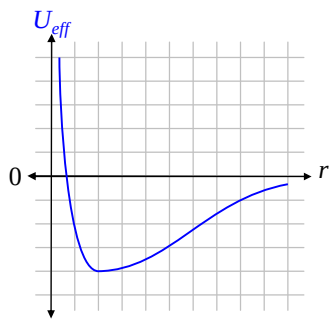

Figure 3.2 Effective potential.

The point is, since we know angular momentum is conserved, we can use it to predict the rotational kinetic energy strictly as a function of r. This eliminates θ from the equations, and reduces the problem to one dimension in r. From the one dimensional viewpoint of the coordinate r, it doesn't matter where the energy goes: it can go into potential energy, or it can go into rotational kinetic energy. The only thing that matters is that both of those are *not* radial kinetic energy, $T_r = \frac{1}{2}m\dot{r}^2$, and *total* energy is conserved.

What's my lagrangian? Since V_{eff} includes some kinetic energy, and the lagrangian distinguishes between kinetic and potential energy, you might wonder, what is the lagrangian for the reduced one-body problem? Recall that the lagrangian is *defined* as the function which, when plugged into the Euler-Lagrange equations, yields the equations of motion. In the reduced one-body problem, the equations of motion are exactly those of a particle in a simple potential. Therefore, the only lagrangian that works is

$$L_{1-body}(r, \dot{r}, t) = T_{1-body} - V_{eff} = \frac{1}{2}m\dot{r}^2 - V_{eff}(r)$$

TBS: Compare the Lagrangian and Hamiltonian views of the reduced (1D) problem in coordinate, r.

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Orbits

To solve for orbits, we must introduce a fixed energy parameter, E, and angular momentum L, as discussed in the "Reduction to 1-Body" section. TBS. The result is:

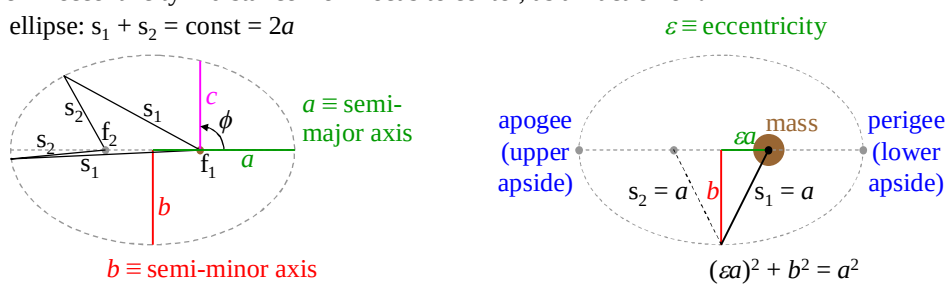
$$r(\phi) = \frac{c}{1+\varepsilon\cos\phi} \quad \textit{where} \quad c, \varepsilon \text{ are orbital parameters .}$$

Other orbital parameters are E, l, a, and b. Any *two* orbital parameters specifies the "standard" orbit, defined as having r_{min} at $\phi = 0$.

The Easy Way to Remember Orbital Parameters

We can easily visualize the relationship between the orbital parameters a, b, and ε, from the diagram below,

where $a \equiv$ semi-major axis
$b \equiv$ semi-minor axis
$c \equiv$ latus rectum
$\varepsilon \equiv$ eccentricity = distance from focus to center, as a fraction of a



(Left) The invariant (1-bounce) distance from focus-edge-focus is $2a$. (Right) Relationship of orbital elements: a, b, and ε.

Recall that an ellipse is defined as the locus of points such that the sum of the distances from the point to the foci is constant: $s_1 + s_2 = \text{const}$ (above left). Choose the point at the end of the major axis, and we see that the constant is $2a$.

(Above right) Choose the point at the end of the minor axis, and $s_1 = s_2 = a$. Then the triangle shown gives

$$(\varepsilon a)^2 + b^2 = a^2 \quad \Rightarrow \quad \varepsilon = \frac{\sqrt{a^2 - b^2}}{a}$$

There two sets of parameters: the *system* parameters, which describe the orbiting bodies, and the force between them; and the *orbital* parameters, which describe a given orbit of the system.

System parameters	$\gamma \equiv$ force constant; $\gamma = Gm_1m_2$, or $\gamma = -k_e q_1 q_2$ $F(r) \equiv -\gamma/r^2$ $M \equiv m_1 + m_2$, $\mu \equiv m_1 m_2/M$
Orbital parameters	$a \equiv$ semi-major axis $b \equiv$ semi-minor axis $c \equiv$ latus rectum $\varepsilon \equiv$ eccentricity = distance from focus to center, as a fraction of a

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	r_{min}, r_{max} ≡ min and max r in orbit l ≡ angular momentum E ≡ *energy*

We start with our fundamental parameters, which we defined in the orbit equation, $r(\phi) = \dfrac{c}{1+\varepsilon\cos\phi}$.

By *geometric* inspection (with no need for dynamics or forces):

$$r_{\min} = \frac{c}{1+\varepsilon}, \qquad r_{\max} = \frac{c}{1-\varepsilon}.$$

From the right triangle containing $f_1 f_2$ and the edge c, and noting that c + hypoteneuse = $2a$:

$$c = a\left(1-\varepsilon^2\right), \qquad \text{from} \qquad (2a-c)^2 = c^2 + (2a\varepsilon)^2.$$

As physical quantities, angular momentum and energy must be related to the force constant, γ.

In the standard coordinates ($\phi = 0$ to the right), any *two* orbital parameters fully defines the orbit.

TBS: Kepler's law about equal areas in equal times results from conservation of angular momentum, and is true for any central potential, not just $1/r$.

Coriolis Acceleration

It turns out that a rotating observer measures two accelerations due to his rotating frame of reference:

(1) centrifugal acceleration, which depends on position, but not velocity, and

(2) Coriolis acceleration, which depends on velocity, but not position.

For now, we call the Coriolis phenomenon an "acceleration" because all masses are accelerated the same (just like gravity). Therefore, we dispense with calling it a "force," which would require us to pointlessly multiply all equations by the accelerated body's mass.

We first derive the Coriolis acceleration from the body displacement relative to the rotating frame. Then, for a more complete view, we re-derive it from the viewpoint of a change of velocity in the rotating frame. We will see that:

The two contributions to Coriolis acceleration are (1) the rotating frame creates a perpendicular component to the original velocity, and (2) a relative velocity of the rotating reference point. The effects are equal and additive, resulting in the factor of two in the Coriolis acceleration formula.

Notation: We consider motion in both an inertial frame, and a frame rotating with respect to it. We use unprimed variables for quantities as measured in the inertial frame, and primed variables for quantities measured in the rotating frame. Note that the vectors $\boldsymbol{\omega}$ and $\mathbf{r}$ are the same in both frames, so there is no $\boldsymbol{\omega}'$ or $\mathbf{r}'$.

Coriolis acceleration from the body displacement: The diagram below illustrates the motion which describes the Coriolis acceleration, drawn in an inertial frame. The angles are exaggerated for clarity, but keep in mind the angles are infinitesimal. The body starts at position $\mathbf{r}$ (below left). In the rotating frame, the body starts with a purely radial velocity, $\mathbf{v}'$. In the inertial frame, its velocity also includes the speed of rotation (upward), so

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}$$

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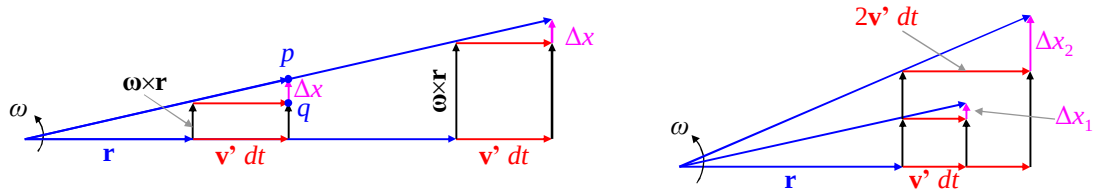

(Left) The change in position Δx between pure radial motion and inertial motion is independent of position. (Right) The change in position quadruples when the time increment is doubled.

However, the body moves with constant velocity in the inertial frame (Newton's 2nd law, with $\mathbf{F} = 0$). After a short time, dt, the body is no longer moving exactly radially. In the inertial frame, it has moved ($\mathbf{v}'$ dt) horizontally, and $(\boldsymbol{\omega} \times \mathbf{r})dt$ upward, to point q. It is now falling behind the rotating frame. Therefore, in the rotating frame, it now has a small component of velocity opposite to that of the rotation. In other words, in the rotating frame, the body has been accelerated downward.

A purely radial motion, throughout the interval dt, would leave the particle at the new radial position p. However, we see that the body has "fallen short" of p by a distance Δx. Δx is the height of a small triangle with (v' dt) as its base, and a narrow angle of (ω dt). The diagram above left shows that this small triangle is independent of the initial radius r. A much larger starting radius results in exactly the same small triangle with the same Δx as its height.

Now, how does Δx increase with dt? Above right: suppose we double our time interval to $2dt$. The base of the small triangle doubles, and the small angle *also* doubles. Therefore, Δx quadruples. This is characteristic of how position varies with a constant acceleration:

$$\Delta x = \left(\frac{1}{2}a\right)(dt)^2$$

Notice that Δx is *independent* of the initial position $\mathbf{r}$. We compute a from simple geometry.

Note that we must carry out our approximations to 2nd order, because our displacement Δx is 2nd order in the small-time parameter dt.

However,

$$\sin(x) = x + \mathcal{O}(x^3),$$

so the usual approximation $\sin(x) \approx x$ is valid to 2nd order. Therefore,

$$\Delta x = v'dt\sin(\omega\, dt) \approx \omega v'(dt)^2 = \left(\frac{1}{2}a\right)(dt)^2 \quad \Rightarrow \qquad a = 2\omega v'$$

Since Δx is opposite $(\boldsymbol{\omega} \times \mathbf{r})$, we can write the Coriolis acceleration as a vector equation:

$$\boxed{\mathbf{a}_{Coriolis} = -2\boldsymbol{\omega} \times \mathbf{v}'}$$

The Coriolis acceleration, by itself, would produce circular motion of the body in the direction opposite the rotation of the reference frame. However, it cannot act by itself, since the body will always be subject to centrifugal forces, too.

Summarizing: A body in a rotating reference frame feels two accelerations, centrifugal and Coriolis:

$$\mathbf{a}_{centrifugal} = \omega^2 r\,\hat{\mathbf{r}} \qquad \textit{and} \qquad \mathbf{a}_{Coriolis} = -2\boldsymbol{\omega} \times \mathbf{v}'$$

Comparing these accelerations, we find (within a given rotating system):

The Coriolis acceleration depends only on velocity in the rotating frame, and not on position.

And, because the Coriolis acceleration is always perpendicular to velocity:

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The Coriolis acceleration does no work.

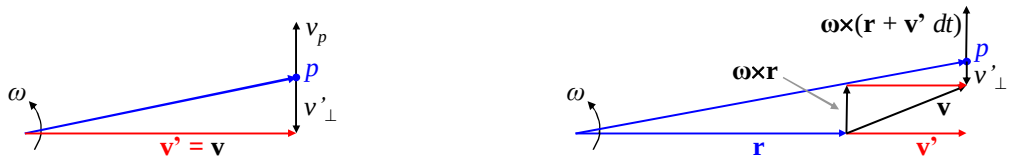
In contrast:

The centrifugal acceleration depends only on radial position, and not on velocity.

And, because the centrifugal acceleration is arbitrarily oriented with respect to velocity:

The centrifugal acceleration can do work and change the system energy.

Coriolis acceleration from velocity-change: It is instructive to consider the Coriolis acceleration from the viewpoint of velocities, rather than changes in position. We now re-derive it as such. Since the Coriolis acceleration is first-order in the velocities, we need keep our approximations only to first order.



Coriolis acceleration as a velocity-change. (Left) Body starts at the origin. (Right) Body starts at arbitrary position.

The full diagram can be confusing, so we first examine the simple case where the body starts at the origin (above left). After a time *dt*, from the inertial frame view, the direction of the radius vector has changed, so that the rotating frame velocity **v'** has a perpendicular component to **r**. The small angle of rotation is $\omega\, dt$, so:

$$v'_\perp = v'\sin(\omega\, dt) \approx \omega v' dt \qquad \text{or, in vector form,} \qquad \mathbf{v}'_\perp = -\boldsymbol{\omega}\times\mathbf{v}'\, dt$$

In addition, the point *p* is fixed on the rotating frame, so it has upward velocity in the inertial frame. In other words, it is "running away from" the velocity vector:

$$\mathbf{v}_p = \boldsymbol{\omega}\times\mathbf{r} = \boldsymbol{\omega}\times\mathbf{v}'dt \qquad \textit{because} \qquad \mathbf{r} = \mathbf{v}'dt$$

Thus there are two contributions to the Coriolis acceleration: (1) the rotating coordinate system causes the initial velocity along **r** to point in a new direction with a downward component in the θ direction; and (2) the point along a fixed radial line is moving upward. These two effects are equal and additive, resulting in the factor of two in the Coriolis acceleration formula. Finally, the body velocity in the rotating frame is the velocity *relative to the point p*; it is the difference between $\mathbf{v}_\perp$ and $\mathbf{v}_p$:

$$d\mathbf{v}' = \mathbf{v}'_\perp - \mathbf{v}_p = -\boldsymbol{\omega}\times\mathbf{v}'\, dt - \boldsymbol{\omega}\times\mathbf{v}'\, dt = -2\boldsymbol{\omega}\times\mathbf{v}'dt \quad \Rightarrow \quad \mathbf{a}_{Coriolis} = \frac{d\mathbf{v}'}{dt} = -2\boldsymbol{\omega}\times\mathbf{v}$$

Now more generally: above right shows the body starting at an arbitrary radius. Initially, in the rotating frame, the body has a purely radial velocity **v'**. This means that in the inertial frame, it is moving upward with velocity

$$\mathbf{v}_{p,initial} = \boldsymbol{\omega}\times\mathbf{r}$$

After the time *dt*, the body has moved to a larger radius, so the reference point *p* moves upward faster than the original ($\boldsymbol{\omega}\times\mathbf{r}$); the new radius is $\mathbf{r} + \mathbf{v}'\, dt$, so the final upward velocity is

$$\mathbf{v}_{p,\,final} = \boldsymbol{\omega}\times(\mathbf{r}+\mathbf{v}'dt) = \boldsymbol{\omega}\times\mathbf{r} + \boldsymbol{\omega}\times\mathbf{v}'dt$$

Comparing initial to final, the upward velocity has *increased* by $\boldsymbol{\omega}\times\mathbf{v}'\, dt$, so the point *p* is again moving upward *away from* the velocity vector with relative velocity $\boldsymbol{\omega}\times\mathbf{v}'\, dt$.

Also as before, the rotating coordinate system creates a component of **v'** in the perpendicular direction:

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$$\mathbf{v}'_{\perp} = -\boldsymbol{\omega} \times \mathbf{v}'\,dt$$

As before, the magnitudes of the upward velocity increase and $\mathbf{v}_{\perp}$ add, giving a factor of 2 in the acceleration formula:

$$d\mathbf{v}' = \mathbf{v}'_{\perp} - \left(\mathbf{v}_{p,\,final} - \mathbf{v}_{p,initial}\right) = -\boldsymbol{\omega} \times \mathbf{v}'\,dt - \boldsymbol{\omega} \times \mathbf{v}'\,dt = -2\boldsymbol{\omega} \times \mathbf{v}'\,dt$$

$$\Rightarrow \qquad \boxed{\mathbf{a}_{Coriolis} = \frac{d\mathbf{v}'}{dt} = -2\boldsymbol{\omega} \times \mathbf{v}}$$

Coriolis acceleration from tangential velocity: So far, we've examined only velocities that are initially purely radial. We now show that a *tangential* velocity results in the same formula for Coriolis acceleration.

Coriolis acceleration for a tangential velocity. (Left) position of body after dt is q. (Right) Velocity vectors at q after dt.

In the diagram above, the angles are exaggerated for clarity, but are really infinitesimal. As always, the rotation of the reference system introduces a perpendicular component to **v'**:

$$\mathbf{v}'_{\perp} = -\boldsymbol{\omega} \times \mathbf{v}'\,dt$$

Now, without any velocity **v'**, the body would have ended up at p. Since the body was moving (in the rotating frame), it actually ends up at q. The velocity of the body in the rotating frame is its velocity *relative to q*. In the inertial frame, q's horizontal velocity is larger leftward than p's by the increased angle θ, which is due to the velocity **v'**:

$$v_q - v_p = r\omega \sin\theta \approx r\omega\left(\frac{v'\,dt}{r}\right) = \omega v'\,dt \qquad \textit{or, in vectors,} \qquad \mathbf{v}_q - \mathbf{v}_p = \boldsymbol{\omega} \times \mathbf{v}'dt$$

This velocity increase is independent of r, because it is proportional to $r\theta$, but θ scales as $1/r$. The final velocity change of the body is then the difference:

$$d\mathbf{v}' = \mathbf{v}'_{\perp} - \left(\mathbf{v}_q - \mathbf{v}_p\right) = -\boldsymbol{\omega} \times \mathbf{v}'\,dt - \boldsymbol{\omega} \times \mathbf{v}'\,dt = -2\boldsymbol{\omega} \times \mathbf{v}'\,dt \qquad \Rightarrow \qquad \boxed{\mathbf{a}_{Coriolis} = \frac{d\mathbf{v}'}{dt} = -2\boldsymbol{\omega} \times \mathbf{v}}$$

Since the Coriolis formula is the same for both radial and tangential velocities, and since the formula is *linear* in those vectors, the Coriolis formula applies to any linear combination of radial and tangential velocities. But *every* velocity can be written as a linear combination of a radius vector and a tangential vector, so the Coriolis formula applies to *all* velocities.

Example of both Coriolis and centrifugal acceleration: Imagine a body at rest in the inertial frame, at some position **r**. In the rotating frame, the body moves in a circle centered at the origin. Therefore, the Coriolis and centrifugal accelerations must produce a centripetal acceleration to maintain circular motion. In the rotating frame, we have velocity and total acceleration:

$$\mathbf{v}' = -\boldsymbol{\omega} \times \mathbf{r}, \qquad \mathbf{a}_{Coriolis} = -2\boldsymbol{\omega} \times \mathbf{v}' = -2\boldsymbol{\omega} \times \left(-\boldsymbol{\omega} \times \mathbf{r}\right) = -2\omega^2 r\hat{\mathbf{r}} \qquad \text{(i.e. inward)}$$

$$\mathbf{a}_{centrifugal} = \omega^2 r\hat{\mathbf{r}} \qquad \text{(outward)}$$

$$\mathbf{a}_{total} = \mathbf{a}_{Coriolis} + \mathbf{a}_{centrifugal} = -\omega^2 r\,\hat{\mathbf{r}} \qquad \text{(inward)}$$

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which is exactly the centripetal acceleration required for circular motion.

Lagrangian for Coriolis Effect

Complicated dynamics problems are often easier to solve with Lagrangian or Hamiltonian mechanics. In a rotating system, then, one needs a lagrangian for the Coriolis acceleration. The Coriolis acceleration is always perpendicular to the velocity, just like the magnetic force. Therefore, we derive the Coriolis lagrangian by analogy with the magnetic force. Since lagrangians produce *forces* (not accelerations), we must write the Coriolis effect as a force, by multiplying by the body mass. Also, to agree with the Lorentz force, we rewrite the Coriolis force as $\mathbf{v} \times \boldsymbol{\omega}$ (rather than $\boldsymbol{\omega} \times \mathbf{v}$), giving:

$$\mathbf{F}_{Coriolis} = 2m\mathbf{v} \times \boldsymbol{\omega} \qquad \leftrightarrow \qquad \mathbf{F}_{Lorentz} = q\mathbf{v} \times \mathbf{B}$$

Therefore, $$m(2\boldsymbol{\omega}) \qquad \leftrightarrow \qquad q\mathbf{B}$$

In other words:

> The angular velocity $2\boldsymbol{\omega}$ acts on a mass
> much like a magnetic field acts on a charge.

Therefore, we can construct a Coriolis vector-potential for the vector ($2\boldsymbol{\omega}$), just as we construct a magnetic vector potential for $\mathbf{B}$:

$$\nabla \times \mathbf{A}_{Coriolis} = 2\boldsymbol{\omega} \qquad \leftrightarrow \qquad \nabla \times \mathbf{A} = \mathbf{B}$$

Then $$L = m\mathbf{v} \cdot \mathbf{A}_{Coriolis} \qquad \leftrightarrow \qquad L = q\mathbf{v} \cdot \mathbf{A}$$

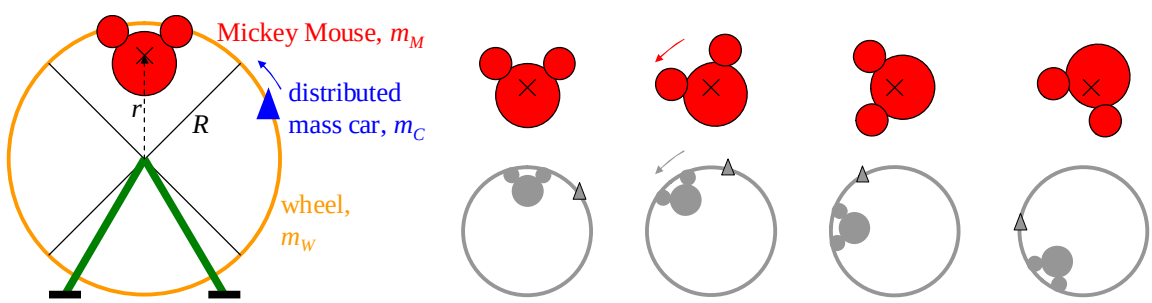
To complete the story for Lagrangian mechanics, we need the potential for the centrifugal force. We get:

$$F_{centrifugal}(r) = m\omega^2 r = -\frac{\partial U(r)}{\partial r}, \qquad \Rightarrow \qquad U_{centrifugal}(r) = -\int_0^r dr\, F(r) = -\frac{1}{2} m\omega^2 r^2 .$$

Mickey Mouse Physics: Parallel Axis

> The parallel axis theorem tells us how to find the moment of inertia of a rigid body around any axis, given its mass and moment of inertia around its center of mass.

Note that the parallel axis theorem does *not* allow us to move from *any* axis to another, only how to move from the center-of-mass axis to any other. [I didn't know this as a 1st-year grad student; I've seen physics professors who don't know it.] Every text book proves this theorem mathematically. We show here how to see it physically. Imagine a Ferris wheel, with a single car on its edge (for simplicity), and a Mickey Mouse attached rigidly to the side:



(Left) A Ferris wheel with 3 significant pieces: the wheel, the car, and Mickey Mouse™.
(Right) Mickey Mouse viewed in isolation as the wheel turns.

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If I were paying the electric bill to run this wheel, I'd want to know its moment of inertia. Since moments of inertia add, we can find the total moment of inertia as:

$$I_{total} = I_{wheel} + I_{car} + I_{Mickey-Mouse}$$

assuming the spokes are negligible. All the mass of the wheel is at the same radius, *R*, so we have the standard formua:

$$I_{wheel} = m_W R^2$$

As the car goes around the wheel, it remains pointing up, so the occupants are comfortable. That is, the car itself doesn't rotate (it just revolves around the wheel center). Therefore, even if the car has a broadly distributed mass, it is only the center-of-mass motion that matters. Hence, the only rotation of the car is its rotation around the wheel, and that is the only moment of inertia that contributes to the total:

$$I_{car} = m_C R^2$$

Mickey is a different story. Mickey Mouse's center of mass is at radius *r*. First, his entire mass rotates in a circle of radius *r*, much like the car. This "global" rotation certainly contributes to his moment of inertia:

$$I_{Mickey-wheel} = m_M r^2$$

In addition, if we look at Mickey's motion about his center-of-mass, we see that Mickey himself also rotates around in a circle (above right). This rotation is simultaneous with, and *in addition to*, his entire mass rotating around the wheel. The car does *not have* this rotation about its center-of-mass. Mickey's self-rotation also contributes to the total rotational inertia, exactly the amount of Mickey's moment of inertia around his center-of-mass. Therefore, Mickey's complete moment of inertia is:

$$I_{Mickey} = m_M r^2 + I_{Mickey-CM}$$

But this is just the parallel axis theorem!

The parallel axis theorem simply sums the "global" rotation of an object's center-of-mass about an axis, with the object's "local" rotation about its own center-of-mass.

Note that if Mickey Mouse is indeed made of 3 circles, we can compute his center-of-mass moment of inertia using, again, the parallel axis theorem on each of the 3 circles.

In unusual cases, we might use the parallel axis theorem in reverse: if we know the moment of inertia about a non-CM axis, and we know the object's mass and distance to the CM-axis, we could compute the moment of inertia about the CM:

$$I_{CM} = I_{non\text{-}CM} - mR^2$$

Moment of Inertia Tensor: 3D

In 3 dimensions, the instantaneous angular momentum of a rigid body is *not* necessarily aligned with the axis of rotation (it *is* when considering 2D planar rotation).

TBS: How can this be??

The moment of inertia tensor relates the angular velocity to angular momentum:

$$\mathbf{M} = \mathbf{I}\boldsymbol{\omega} \qquad or \qquad M_i = \sum_{j=1}^{3} I_{ij}\omega_j$$

(add a real picture of the vectors here).

[In fact, the moment of inertia tensor is a Cartesian tensor, not a true Riemannian tensor, because it involves finite-displacements of the mass points from the rotation axis.]

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Rigid Bodies (Rotations)

Moment of Inertia: $$I_{ij} = \sum_p m_p \left(\delta_{ij} r_p{}^2 - x_{pi} x_{pj}\right) = \sum_p m_p \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix}$$

$\forall i, j, k:\ I_i + I_j \ge I_k$ For flat object in $x-y$: $I_z = I_x + I_y$ [L&L 32.9+ p100]

Parallel-axis: $I_{ij} = I_{ij}^{cm} + M\left(a^2\delta_{ij} - a_i a_j\right)$ [F&W26.19 p140, L&L 32.12 p101]

Euler eqs:

$$\begin{aligned} I_1\dot{\omega}_1 &= \omega_2\omega_3\left(I_2 - I_3\right) & & I_1\dot{\omega}_1\omega_1 = \omega_1\omega_2\omega_3\left(I_2 - I_3\right) \\ I_2\dot{\omega}_2 &= \omega_3\omega_1\left(I_3 - I_1\right) & \Rightarrow\quad & I_2\dot{\omega}_2\omega_2 = \omega_2\omega_3\omega_1\left(I_3 - I_1\right) \\ I_3\dot{\omega}_3 &= \omega_1\omega_2\left(I_1 - I_2\right) & & I_3\dot{\omega}_3\omega_3 = \omega_3\omega_1\omega_2\left(I_1 - I_2\right) \end{aligned}$$

Rotating Frames

Recall that a **reference frame** is an infinite set of points whose distances to an observer remain constant. One reference frame may rotate with respect to another reference frame. Frames in which Newton's 1st and 2nd laws are obeyed are called **inertial frames**. If two reference frames are rotating with respect to each other, then at most one of those frames can be inertial.

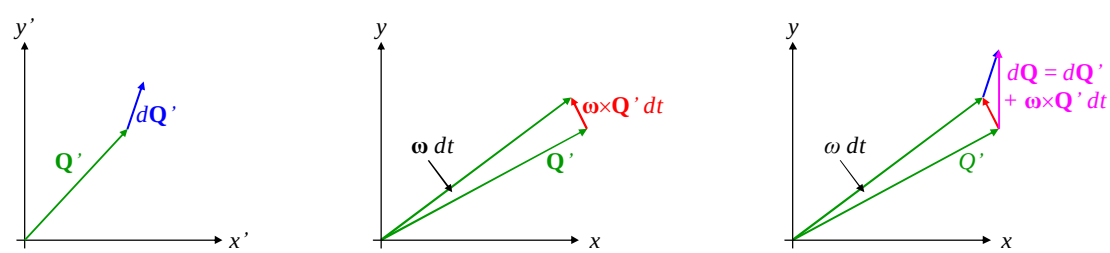
We now consider the case of an inertial frame (the "fixed frame"), and a (non-inertial) frame rotating with respect to the inertial frame. Consider a continuously changing vector in the fixed frame, **Q**(*t*). For any such vector, its time derivative is also a vector:

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{fixed} = \left(\frac{d\mathbf{Q}}{dt} + \boldsymbol{\omega}\times\mathbf{Q}\right)_{rotating}.$$ [compare to M

Writing **Q** for the vector in the fixed frame, and **Q'** for the vector in the rotating frame, we get:

$$\frac{d\mathbf{Q}}{dt} = \frac{d\mathbf{Q}'}{dt} + \boldsymbol{\omega}\times\mathbf{Q}'.$$

Note that in the $\boldsymbol{\omega}\times\mathbf{Q}$ term, if the vector **Q** is *not itself a time derivative* (e.g., an electric field), then it is an instantaneous vector (a geometric object, an "arrow in space") and is the same in either frame. Hence, it does not need qualifying as "fixed" or "rotating."



Change of vector in the rotating frame | Change of vector in the fixed frame, due to rotation | Total change of vector in the fixed frame

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The uncorrected equation in [M&T] is this: $\left(\frac{d\mathbf{Q}}{dt}\right)_{fixed} = \left(\frac{d\mathbf{Q}}{dt}\right)_{rotating} + \boldsymbol{\omega} \times \mathbf{Q}$. The problem with this equation is that the $\boldsymbol{\omega}\times\mathbf{Q}$ term is ambiguous: is the Q in this term from the fixed frame or the rotating frame? [M&T] incorrectly hint that this is irrelevant *in all cases*, because this is an abstract vector equation, independent of reference frame. They imply that the ambiguous **Q** is therefore a vector at an instant in time (it's just an arrow in space), and independent of reference frame (as described above). But this is wrong.

For things like the Coriolis acceleration, **Q** is *itself* a time derivative: $\mathbf{v} = d\mathbf{r}/dt$, the velocity. There is a big difference in both magnitude and direction between **v**, the velocity in the fixed frame, and **v'**, the velocity in the rotating frame. In other words, **v** and **v'** are *not* the same vector written in two different frames; they are truly different vectors [S&C2 p151b]. The diagram above shows that the component of acceleration $(d\mathbf{Q}/dt)_{fixed}$ due to rotation is $\boldsymbol{\omega}\times\mathbf{v}'$, and *not* $\boldsymbol{\omega}\times\mathbf{v}$.

Rotating Bodies vs. Rotating Axes

To rotate an axis (say the *z* axis) to any position requires two parameters, say θ and ϕ. To rotate a rigid body requires 3 parameters, say θ, ϕ, and ψ. The reason for the difference is that rotating an axis about itself produces no change, and is hence unnecessary. For example, consider rotations of a baseball bat with a label on the side. Starting from anywhere, it takes a θ and ϕ rotation to align the axis of the bat to an arbitrary axis. It then takes a third rotation, ψ, to rotate the label to the desired direction. If the baseball bat is replaced by an axis, then the last rotation is meaningless; axes have no ψ rotation.

Arbitrary rotations of an axis require two parameters, θ and ϕ.
Arbitrary rotations of a rigid body require 3 parameters, θ, ϕ, and ψ.

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4 Shorts

Here is a collection of short topics which can be read independently of any sequence of topics.

Friction

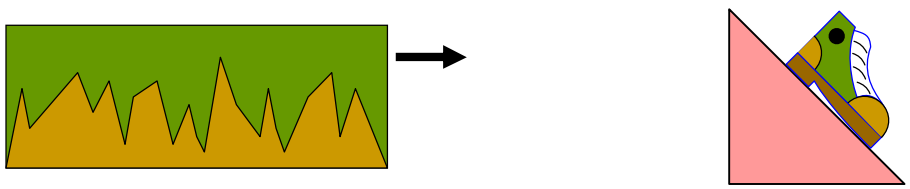
Static Friction

When dealing with friction, there are two distinct cases that must be considered: with slipping, and without. Without slipping, the friction force becomes as big or small as it needs to be to prevent slipping. But there's a limit: friction is only so strong. It's maximum value is given by

$$F_{\max} = \mu_s N \qquad \textit{where} \quad \mu_s \equiv \text{coefficient of static friction}$$
$$N \equiv \text{perpendicular (aka normal) force}$$

Many books incorrectly state that the coefficient of friction must be < 1. This is false. Consider dragging a piece of Velcro across its mating piece.
The coefficient of friction is clearly greater than one.

Other examples include dragging tape across a desk (sticky side down), or any rough surfaces with deep thin spikes in their microscopic surface. Or macroscopic surface, since there's nothing that needs be microscopic about friction (though it *usually* is).



(Left) Microscopic (or macroscopic) view of how to make a coefficient of friction $\mu > 1$.
(Right) Simple test for $\mu > 1$.

There's a simple way to test if $\mu_s > 1$: tilt the surfaces to a 45° angle. If the surfaces don't slip, $\mu > 1$.

Often, in a given problem, we don't know ahead of time whether there will be slipping or not. Therefore we must figure it out. Usually, we must solve the equations first *assuming* no slipping, and compute what frictional force F is required to enforce no slipping. If it is within F_{max}, then there is no slipping. If it exceeds F_{max}, then slipping does occur. We must then replace F with F_{max} in our equations, and solve them again to get the final answer. The example below illustrates this, and how, without slipping, the unknown friction F is compensated by a kinematic relation equation, and with slipping, F is directly calculated.

Kinetic Friction

Kinetic friction is the friction force when surfaces are slipping. It's coefficient μ_k is always < μ_s. You can feel that $\mu_k < \mu_s$ when you push on something that's not moving, and it suddenly "breaks free" and starts moving with less push needed. You needed more push when it was static because $\mu_s > \mu_k$. μ_k may also be > 1. The simple test for $\mu_k > 1$ is similar to that for μ_s: tilt the surface to 45° and give them a push, to start them slipping. If they stop slipping, $\mu_k > 1$.

Mathematical Formula for Static and Kinetic Friction

Rarely, it's handy to be able to write static and kinetic friction as mathematical formulas. Of course, they are non-linear, which is why friction problems can be difficult. If x is the coordinate in which N is the normal (perpendicular) force:

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(Left) static friction. (Right) kinetic friction.

Static friction: Static friction is defined as friction when there is no slipping. In other words:

$$v \equiv \dot{x} = 0 \qquad \text{(static friction)}$$

F_x is a velocity constraint force, constraining $v = 0$. As a *constraint*, F_x must be *solved for*. However, the *maximum* force static friction can exert is:

$$F_{s.\max} = \mu_s N \qquad \textit{where} \quad \mu_s = \text{coefficient of static friction}$$

Kinetic friction: Kinetic friction is defined as friction when there *is* slipping (particle is moving). Then the force of friction is *determined* by the normal force, according to:

$$F_x = -\mu_k N \operatorname{sgn}(v) \qquad \textit{where} \quad v \equiv \text{velocity of slipping}$$

$$\mu_k \equiv \text{coefficient of friction}$$

$$\operatorname{sgn}(v) \equiv 1,\ v > 0; \quad \operatorname{sgn}(v) = -1,\ v < 0$$

Note that friction always acts *opposite* to the slipping, indicated by the negative sign. In 3 dimensions, we can write the force of friction as:

$$\mathbf{F} = -\mu_k N \hat{\mathbf{v}} \qquad \textit{where} \quad \hat{\mathbf{v}} \equiv \text{unit vector is direction of the velocity of slipping .}$$

Example: Friction With and Without Slipping

Consider a spool with a central shaft of radius r, and larger wheels of radius R (figure below). Around the shaft is wound a string, on which we pull with tension T at an angle θ with the horizontal.

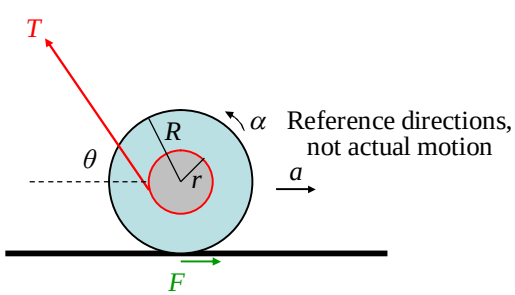

Figure 4.1 The arrows for α and a show the *reference* directions, not the actual motion. In fact, when rolling without slipping, one of them must be negative.

Under all circumstances, from *Force* = *ma* and torque $\tau = I\alpha$, we have:

$$(1) \quad ma = F - T\cos\theta \qquad \textit{and} \qquad (2) \quad I\alpha = FR - Tr$$

where we choose to measure torques about the central axis. We consider two questions, (a) and (b):

(a) We are given T, m, I, θ, r, R, and μ (for simplicity, fixed for both static and kinetic friction). We seek a, α, and F. Therefore, we need 3 equations. Equations (1) and (2) provide two of them. For the 3rd, we must consider the cases of rolling and slipping separately, because the frictional force is either unknown (rolling), or known directly (slipping). When rolling, the unknown F is augmented by an additional

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kinematic constraint equation that relates a and α. The transition from rolling to slipping is given by the maximum friction equation:

$$F \le F_{\max} = \mu N = \mu\left(mg - T\sin\theta\right)$$

We will see that for any angle θ, there exist tensions that allow rolling (or at least, no slipping), and higher tensions that demand slipping.

Case 1: Rolling (really, no slipping)

First, we must *assume* no slipping, and test our result for consistency at the end. When rolling (or staying still without slipping), the linear acceleration is related to the angular acceleration by

$$a = -R\alpha\ ,$$

and F must be solved for. (We derive the above by considering the instantaneous rotation of the spool about its point of contact with the ground.) Eliminating first a, then α, from our (1) and (2) yields:

$$(3)\quad m\left(-R\alpha\right) = F - T\cos\theta \qquad \textit{and} \qquad I\alpha = FR - Tr \qquad \rightarrow \qquad (4)\quad \alpha = \left(FR - Tr\right)/I$$

$$-m\frac{R\left(FR - Tr\right)}{I} = F - T\cos\theta$$

$$T\cos\theta + \frac{mTRr}{I} = F + \frac{mR^2F}{I} = F\left(1 + \frac{mR^2}{I}\right)$$

$$F = \frac{T\left(\cos\theta + mRr/I\right)}{1 + mR^2/I}$$

If $F < F_{max}$, then our assumption of no slipping is valid. Then, substituting F into (1) and (2) gives a and α. If not, then we have Case 2: slipping.

Case 2: Slipping

With slipping, a and α are now independent (the kinematic constraint doesn't apply). But this only happens when friction is at its maximum, so we find F directly (assuming for simplicity (though unrealistically) that $\mu_s = \mu_k$):

$$F = F_{\max} = \mu\left(mg - T\sin\theta\right)$$

We then plug directly into (1) and (2) to solve for a and α.

What if we start with the reverse assumption: assume slipping, and check for consistency? What is inconsistent if we start by assuming slipping, and there really is no slipping? The friction force will end up doing more work than the tension puts into the system, and therefore being a *source* of energy. This is not possible, but is (in my opinion) harder to check than assuming no slipping, and simply checking if the required friction force exceeds the maximum available.

(b) If we pull gently enough (T small), there will be no slipping. As we increase T, at some point, the wheel starts to slip. For given other parameters, at what T does slipping begin?

We start with our equation for F in the case of no slipping from part (a). F is proportional to T, and the maximum F is

$$F_{\max} = \mu N = \mu\left(mg - T\sin\theta\right)$$

Note that as T increases, the required F for no slipping increases, and *also* the maximum F decreases. When the F required for no slipping equals the maximum F, we have the transition tension:

$$F_{\max} = \mu\left(mg - T\sin\theta\right) = T\frac{\cos\theta + mRr/I}{1 + mR^2/I} \quad\Rightarrow\quad T = \mu mg\left(\frac{\cos\theta + mRr/I}{1 + mR^2/I} + \mu\sin\theta\right)^{-1}$$

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Rolling Friction

In reality, when something rolls without slipping, there is usually still some energy loss due to the rolling interface. This loss is called **rolling friction**, or rolling resistance, or rolling drag. There are several reasons for this: elastic deformation of the wheels, or the surface beneath them, converts some of the energy to heat (more so with rubber tires than steel wheels). Imperfections in the circular wheels or flat surface cause some amount of actual scraping as it rolls. Differences in the wheel sizes on each side of an axel force some slippage of one or both wheels, which dissipates energy to friction. Imperfections increase bouncing, and therefore vibration, which carries away energy. Stickiness between the surfaces can also contribute. The coefficient of rolling friction is typically much less than μ_k, the coefficient of kinetic (sliding) friction. The drag force from rolling friction is computed similarly to static and kinetic friction:

$$F = \mu_r N \qquad \textit{where} \quad \mu_r \equiv \text{coefficient of rolling friction}$$

μ_r for a new car tire is typically ~ 0.01 [en.wikipedia.org/wiki/Rolling_resistance].

Drag

Drag is a kind of friction when a body moves through a fluid (liquid or gas).

Drag differs from kinetic friction in that drag depends on velocity.

Specifically,

$$\mathbf{F} = -a\mathbf{v} - bv^2\hat{\mathbf{v}} \qquad \textit{where} \quad \mathbf{v} \equiv \text{velocity of moving body;} \;\; v^2 \equiv \mathbf{v}\bullet\mathbf{v}$$

Like most friction, drag acts opposite to the motion.

Damped Harmonic Oscillator

An unforced damped harmonic oscillator has equation of motion:

$$\ddot{x}(t) + 2\beta\dot{x}(t) + \omega_0{}^2 x(t) = 0 \qquad \textit{where} \quad \omega_0 \equiv \text{undamped oscillation frequency}$$
$$\beta \equiv \text{damping factor}$$

This is a linear differential equation with constant coefficients; it's solutions are therefore exponentials (possibly complex, which are equivalent to sines and cosines). This section assumes the reader has been introduced to damped harmonic oscillators, and the method of solving their differential equations.

Since the differential equation is linear, the response of the system to "excitations" (things that make it move) is linear. The excitations of a system are the initial conditions plus any forcing functions on the system. In this case, the right hand side is 0, so there are no forcing functions. Therefore, linearity implies:

If	$x^1(0), \dot{x}^1(0)$	produce the response	$x^1(t)$
and	$x^2(0), \dot{x}^2(0)$	produce the response	$x^2(t)$
then	$kx^1(0) + x^2(0), k\dot{x}^1(0) + \dot{x}^2(0)$	produce the response	$kx^1(t) + x^2(t)$

The solutions to the above equation come in 3 distinct forms, depending on how β compares to ω_0.

$\beta < \omega_0$	**underdamped**: the system oscillates with exponentially decaying amplitude
$\beta = \omega_0$	**critically damped**: the system decays without oscillation the most rapidly possible
$\beta > \omega_0$	**overdamped**: the system decays without oscillation

You can think of $\beta < \omega_0$ as "more oscillatory than damped," so it's underdamped and oscillates. Then $\beta > \omega_0$ is "more damped than oscillatory," so it's overdamped and doesn't oscillate.

Sometimes we refer to the **damping ratio** defined as $\zeta \equiv \beta/\omega_0$. Then $\zeta < 1$ is underdamped, $\zeta = 1$ is critically damped, and $\zeta > 1$ is overdamped.

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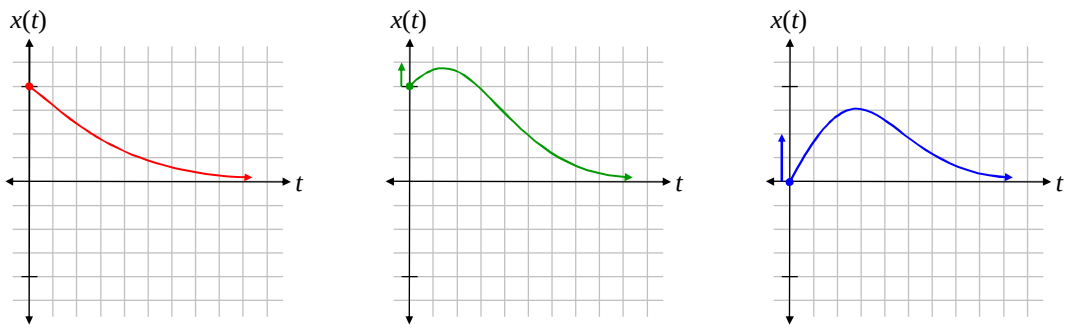
Critical Condition

This section explores some properties of critically damped oscillators. By definition, a **critically damped oscillator** has $\beta = \omega_0$. The characteristic equation therefore has two equal real roots of $-\beta$. Since the roots are real, the system does not oscillate. What does the free (unforced) response of a critically damped system look like? Does it cross the axis, or not? The answer depends on the initial conditions. Let us consider 4 possible initial conditions:

1. Displaced, and released with zero speed.
2. Displaced, and released with velocity away from 0 (increasing the displacement)
3. Positioned at equilibrium, but released with some velocity
4. Displaced, and released with velocity toward 0 (toward equilibrium)

We can qualitatively describe the first three of these without solving any equations. The last one requires some simple math.



Critically damped oscillator response for three cases of initial position and velocity..

Case 1: Displaced, and released with zero speed (above left): Since we know a critically damped system doesn't oscillate, it must be that if I let it go from some displacement, but with zero initial speed, it will not cross the axis. For if it did, then it must stop at some point on the other side. Then, it will be starting from some (smaller) displacement, and zero speed, much like it was a moment ago. If it crossed once, it must now cross again, because the system is linear and the response is proportional to the displacement. But if it crosses twice, it will again reach some highest point, and again be starting from some displacement with zero speed. This would go on forever, and would be oscillation, contradicting our prior knowledge that it does not oscillate.

Case 2: Displaced, and released with velocity increasing the displacement (above center): At some point, the restoring force will stop the increasing motion. Then we have the system displaced, and with zero speed, which is Case 1. Therefore, the system does not cross the axis.

Case 3: Positioned at equilibrium, but release with some velocity (above right): Then the system will reach a maximum displacement, when the restoring force has stopped the increasing motion. The system is now displace, and with zero speed. Again, we are back to Case 1. The system does not cross the axis.

Case 4: Displaced, and with initial velocity toward equilibrium. For tiny velocities, this is only marginally different from Case 1, so we suspect that the system still will not cross the axis. We prove it by noting that for Case 1, sometime after release, the system has displacement with velocity toward equilibrium, but it still does not cross. For larger velocities, can it cross? For this, we must solve equations. By definition, a critically damped system has $\beta = \omega_0$, and therefore has duplicated roots of the characteristic equation:

$$r_1 = r_2 = -\beta \qquad \Rightarrow \qquad x(t) = Ae^{-\beta t} + Bte^{-\beta t} = \left[A + Bt\right]e^{-\beta t}$$

where A and B are determined from the initial conditions, as follows:

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$$\dot{x}(t) = -\beta A e^{-\beta t} + B\left(e^{-\beta t} - \beta t e^{-\beta t}\right) = (B - \beta A)e^{-\beta t} - B\beta t e^{-\beta t} = \left[(B - \beta A) - B\beta t\right]e^{-\beta t}$$

$$\textit{Then} \qquad x(0) = A, \qquad \dot{x}(0) = B - \beta A \quad \Rightarrow \quad B = \dot{x}(0) + \beta x(0)$$

Plugging into the solution for $x(t)$, we get the complete solution for the given initial conditions (ICs):

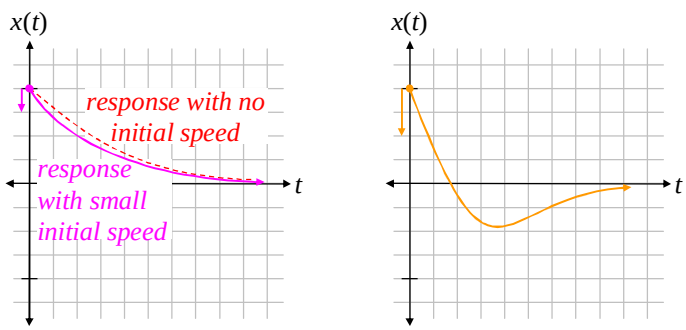
$$x(t) = x(0)e^{-\beta t} + \left(\dot{x}(0) + \beta x(0)\right)te^{-\beta t} = \left[x(0) + \left(\dot{x}(0) + \beta x(0)\right)t\right]e^{-\beta t}$$

To cross the axis, there must be a time $t > 0$ when $x(t) = 0$. Thus

$$0 = x(0) + \left(\dot{x}(0) + \beta x(0)\right)t \qquad \Rightarrow \qquad t = -\frac{x(0)}{\dot{x}(0) + \beta x(0)}$$

WLOG (without loss of generality), we take $x(0)$ positive, and therefore x-dot(0) is negative. For the zero crossing time to be positive, the above denominator must be negative:

$$\dot{x}(0) + \beta x(0) < 0 \qquad \textit{or} \qquad \dot{x}(0) < -\beta x(0)$$



Critically damped oscillator response for two different initial speeds toward the axis: (Left) Speed < $\beta x(0)$. (Right) Speed > $\beta x(0)$.

In words, if the initial speed is large enough toward 0, the system will cross the axis only once. At some point on the other side, it will stop. It will then have displacement but no velocity, and we are back to Case 1. It will asymptotically approach the axis, but not cross it again.

Pressure

The forces resulting from interior pressure can sometimes be surprising. Consider a square-cross-section cylinder, filled with a gas under pressure (below). (Some cargo airplanes have roughly this cross-section.) We can find the force exerted on side R, tending to rip the metal walls along the edge, by multiplying the pressure times a cross-section taken vertically. This is:

$$F = PA$$

Force on a square tube under pressure: the force along the walls is different than at the corners.

By choosing a different cross-section, though, we get a different answer. The force acting to separate the cylinder at its corners is different than that acting to rip its walls. If we choose a diagonal (above, right) as our cross-section, we find the area is larger by $\sqrt{2}$, and therefore so is the force: $F = \sqrt{2}PA$.

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We can compute this second result a different way, by taking the components of the forces on walls T and R that lie along the direction of the force acting to separate the cylinder at its corners. Wall T has force *PA*, and the component (in red) in the corner-separating direction is $(1/\sqrt{2})PA$. But wall R has a similar component, and adding the two gives the total force, which is again $F = \sqrt{2}PA$.

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5 Intermediate Mechanics Concepts

Generalized Coordinates

Generalized coordinates are the essence of analytic mechanics. Instead of the usual (x, y, z) or other simple coordinates, **generalized coordinates** measure displacements in units natural to a given problem: they may be angle, distances, or other numbers measured with respect to arbitrary references, such as arc-length along some path. The **configuration-space** (see below) defines the position of all points in the system, and the generalized coordinates are usually labeled $q_i = \{q_1, \ldots q_n\}$.

This leads to **generalized force**, which is the quantity which satisfies the work equation in generalized coordinates [F&W 15.4 p54m]:

$$\delta W = Q_i\, dq_i, \qquad \textit{where} \quad Q_i \equiv \textit{generalized force, sometimes written as} \qquad Q_i = \frac{\partial W}{\partial q_i},$$

but this latter form is worrisome, since work is an inexact differential (ignore this if you don't understand inexact differentials). Generalized forces also satisfy the momentum equation [ref??]:

$$F = \frac{dp}{dt} \equiv \dot{p} \qquad \Rightarrow \qquad Q_\sigma = \dot{p}_\sigma$$

I Need My Space: Configuration Space, Momentum Space, and Phase Space

Configuration space is the space of all possible values of generalized position coordinates; it is n dimensional, where n is the # of generalized coordinates of the system. Note that configuration space is an abstract space, not (in general) physical space. If there are k constraints between the coordinates, then there are only $(n - k)$ independent degrees of motion in the system, and the allowed points in configuration space form an $(n - k)$-dimensional subspace of configuration space.

Momentum space is the n-dimensional space of all possible values of generalized momenta. Again, constraints may restrict the state of the system to a subspace of momentum space.

Phase space is the aggregate of configuration and momentum space: it is the $2n$-dimensional space of all possible position and momentum values. [Mathematically, phase-space is the tensor product of configuration-space and momentum-space.] Summarizing:

Configuration space:	$(q_1, q_2, \ldots q_n)$
Momentum space:	$(p_1, p_2, \ldots p_n)$
Phase space:	$(q_1, p_1, q_2, p_2, \ldots q_n, p_n)$

Choosing Generalized Coordinates, and Finding Kinetic Energy

There are no definitive rules for the "best" choice of generalized coordinates. Usually:

We choose coordinates that are the minimum needed to specify the motion,
and which reflect the symmetries of the system.

Symmetries include the form of the potential energy, and any constraints. Sometimes, we choose more coordinates than are needed, and impose separate constraints among them (see Motion With Constraints, later).

To write the lagrangian in generalized coordinates, we need to express the kinetic and potential energies in those coordinates. Since the symmetry of the potential energy is a factor in our choice of coordinates, the potential energy is usually fairly easy to write in terms of the coordinates.

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However, kinetic energy is often harder. The generalized coordinates might have complicated relationships to Cartesian coordinates, and they may be time dependent. A very common procedure for finding the kinetic energy is to write the position of each particle in Cartesian coordinates, as a function of the generalized coordinates, and possibly time:

$$x = x(q_i, t) \qquad y = y(q_i, t) \qquad z = z(q_i, t)$$

Then the magnitude of the velocity squared is obtained by taking time derivatives of the Cartesian coordinates, and summing their squares:

$$v^2(q_i, t) = \dot{x}^2(q_i, t) + \dot{y}^2(q_i, t) + \dot{z}^2(q_i, t) \qquad \Rightarrow \qquad T = \frac{m}{2}\left[\dot{x}^2(q_i, t) + \dot{y}^2(q_i, t) + \dot{z}^2(q_i, t)\right]$$

A simple example: Let us compute the lagrangian for a simple pendulum, in the generalized coordinate θ, measured as angular displacement from the vertical. In this case, both kinetic and potential energy are most easily expressed in Cartesian coordinates x and y. Therefore, we need to convert from θ to (x, y) for both T and U. [You might wonder, then, "Why didn't we use (x, y) coordinates in the first place?" The reason is that using θ requires only one generalized coordinate, whereas using (x, y) requires two coordinates and a constraint equation. Try it, and you'll quickly see that (x, y) coordinates are nearly intractable.] For potential energy, we have:

$$U(x, y) = mgy \qquad \textit{and} \qquad y = Y(\theta, t) = -l\cos\theta \qquad \Rightarrow \qquad U(\theta) = -mgl\cos\theta$$

For kinetic energy:

$$x(\theta, t) = l\sin\theta, \qquad \dot{x}(\theta, t) = \left(l\cos\theta\right)\dot{\theta}$$

$$y(\theta, t) = -l\cos\theta \qquad \dot{y}(\theta.t) = \left(l\sin\theta\right)\dot{\theta}$$

$$T = \frac{m}{2}\left[\dot{x}^2(\theta, t) + \dot{y}^2(\theta, t)\right] = \frac{m}{2}l^2\dot{\theta}^2\left(\cos^2\theta + \sin^2\theta\right) = \frac{m}{2}l^2\dot{\theta}^2$$

We see that the final v^2 could also have been deduced from the definition of a "radian." Note that we *frequently* need to use trigonometric identities to simplify our final results in terms of our generalized coordinates. Know your sum and difference angle identities (see Formulas section).

Manifold mathematicians will notice that in computing the kinetic energy, we have actually computed the metric field for our coordinates. Note that the metric field is only a true tensor in a coordinate basis, and so the metric computed here is generally not a true tensor field.

D'Alembert's Principle

[Section under construction.] D'Alembert's principle is *the* fundamental principle of analytical mechanics, according to Cornelius Lanczos' *Variational Principles of Mechanics* [Lan p77t]. [Lan ch 3] has an interesting discussion of the progression from Newton's laws, to d'Alembert's principle. D'Alembert's principle is equivalent to Hamilton's principle, and other lesser-known principles of Euler, Lagrange, and Jacobi. Therefore, it is the "*only* postulate of analytical mechanics," as Lanczos sees it.

D'Alembert's principle can be integrated with respect to time to get Hamilton's principle [Lan p11-3].

Hamilton's principle is valid only when d'Alembert's principle is valid.

[Lan] points out specifically how action-minimizing principles now describe all fields of physics, far beyond what Lagrange could have anticipated.

D'Alembert's principle of virtual work (1742) attempts to "reduce" dynamics problems to "statics." Essentially, we rename the rate of change of momentum as a "force": the **inertial force**. Then the sum of all the "forces" equals zero. For example:

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$$\mathbf{F} = m\mathbf{a} = \frac{d\mathbf{p}}{dt} \qquad \rightarrow \qquad \mathbf{F} - \dot{\mathbf{p}} = 0 = \mathbf{F} + \mathbf{F}_e = \sum_{i=1}^{\#\ particles} \mathbf{F}_i = 0, \qquad where \quad \mathbf{F}_e \equiv -\dot{\mathbf{p}}$$

This looks like a statics problem. Similarly for rotations, we relate torque and angular momentum:

$$\mathbf{T} = I\ddot{\boldsymbol{\theta}} = \frac{d\mathbf{L}}{dt} \qquad \rightarrow \qquad \mathbf{T} - \dot{\mathbf{L}} = 0 = \mathbf{T} + \mathbf{T}_e = \sum_{i=1}^{\#\ particles} \mathbf{T}_i = 0, \qquad where \quad \mathbf{T}_e \equiv -\dot{\mathbf{L}}$$

The inertial force originates from the center-of-mass of the system; the inertial torque is a moment, and can act about any arbitrary point (in fact, it's the same around *all* points).

In a static system, the sum of the forces is zero. Therefore, the gradient of the total potential, which is the sum of the forces, is zero:

$$\sum \mathbf{F} = -\nabla V = 0$$

For a small displacement of the system, the work done to achieve the displacement equals the change in potential energy, which is simply $\nabla V \cdot \delta\mathbf{r}$:

$$\delta W = \nabla V \cdot \delta\mathbf{r} = \sum \mathbf{F} \cdot \delta\mathbf{r} = 0 \qquad \text{(infinitesimal displacement } \delta\mathbf{r}\text{)}$$

By calling $m\mathbf{a}$ a "force", we have one form of d'Alembert's principle: Constraint forces do no work on virtual displacements. Why??

Like all good variational principles, according to [Lan], d'Alembert's principle applies to generalized coordinates, not just rectangular ones. In addition, d'Alembert's principle applies for both holonomic (integrable) and non-holonomic constraints, making it more general than Hamilton's stationary-action principle. [Liu] disputes this claim.

[Liu] states that d'Alembert's principle applies to velocity constraints only when the velocity terms of the constraint equations are homogeneous (of any order) in the velocities. In other words, when we can write the constraint equations as:

$$g(q_i, \dot{q}_i, t) = f(q_i, t) + s(q_i, \dot{q}_i, t) \qquad where \quad s(q_i, \dot{q}_i, t) \text{ is homogeneous in the } q_i$$

This is a very large class of velocity constraints, which covers most practical situations.

Lagrangian Mechanics

Introduction

The importance of Lagrangian physics cannot be overemphasized:

All of modern (microscopic) physics, and much macroscopic physics,
can be described by the principle of stationary-action.
To describe different phenomena, we need only find the right lagrangian.

Lagrangian mechanics simplifies calculations through two principles: (1) generalized coordinates, and (2) Hamilton's principle, i.e. the principle of stationary action. Machines, including all rigid bodies, always incorporate parts that constrain each other's motion. Such constraints reduce the independent degrees of freedom from the $3N$ of N point particles. Generalized coordinates allow specifying the positions of all parts of a system, i.e. its **configuration**, usually with a minimum number of coordinates. In some cases, when additional constraints between coordinates are desirable or necessary, they can usually be included in the Lagrangian method.

Many people mistakenly believe that Lagrangian mechanics is based on conservation of energy and/or momentum. Neither of these is true.

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In fact, for macroscopic systems, Lagrangian mechanics applies equally well to systems that conserve energy and/or momentum, and those that don't. Furthermore, Lagrangian mechanics provides simple methods for determining whether energy or momentum is conserved in a given system, but either way, Lagrangian mechanics provides the correct equations of motion. Microscopic physics always conserves energy, so there is no need to account for friction, as there is in macroscopic physics. However, some frictional forces can be added into the general Lagrangian framework, though they do not follow the least-action principle.

In simplest form, Lagrange's equations can be thought of as the freshman physics concept that rate-of-change of momentum equals force: $dp/dt = F$, though extended to generalized coordinates. For non-relativistic, non-magnetic mechanics,

$$L(q,\dot{q},t) = T(\dot{q}) - V(q,\dot{q},t) \Rightarrow \qquad F = -\frac{\partial V}{\partial q} = \frac{\partial L}{\partial q}$$

$$\textit{Lagrange's equations}: \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \qquad \text{(aka Euler-Lagrange equations)}$$

$$\textit{Can be written}: \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$$

$$\textit{Defining} \qquad p \equiv \frac{\partial L}{\partial \dot{q}} \quad \Rightarrow \quad \frac{d}{dt}p = \frac{\partial L}{\partial q} = F$$

Note that if some of the forces are not included in the lagrangian (such as non-potential forces, e.g. friction), we can include them on the right hand side of Lagrange's equations, essentially by using :

$$\frac{dp}{dt} = \sum_{i=1}^{\#\,forces} F_i = \sum_{j=1}^{\#\,Lagrangian\ forces} F_j + \sum_{k=1}^{\#\,non-Lagrangian\ forces} F_k \quad \Rightarrow \quad \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \sum_{k=1}^{\#\,non-Lagrangian\ forces} F_k \,.$$

What is a Lagrangian?

Circular though it may seem,

> The lagrangian of a system is *defined* as the function (typically of q, q-dot, and t) which, when put into the Euler-Lagrange equations, yields the equations of motion of the system.

There is no general method for doing the reverse: finding a lagrangian from the equations of motion [Gol??]. However, [J&S] develop the lagrangian for a charged particle in a magnetic field by first finding several conditions such a lagrangian must satisfy, and then one needs only a small amount of "guess and check" to finish. As we will show, there are infinitely many lagrangians for any system, which all give the same EOMs. Also, different systems can have the same lagrangian (a hoop rotating flat about a point is the same as a pendulum in gravity??). (There are some unusual examples where a lagrangian gives the correct equations of motion, but appears to be the wrong lagrangian [ref??], but we need not address that now.)

Equivalently (and still somewhat circularly), the lagrangian can be defined as the function whose time integral gives the action of the motion.

We will see that:

> The value of the lagrangian function is independent of the chosen generalized coordinates, since it is defined by terms which are coordinate independent.
> However, the lagrangian *does* depend on the chosen zero of potential energy.

For example, the nonrelativistic nonmagnetic lagrangian is $T - V$, which is independent of coordinate choice:

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$$L(q_i, \dot{q}_i, t) = T(q_i, \dot{q}_i, t) - V(q_i, \dot{q}_i, t), \qquad \text{(general non-relativistic lagrangian)}$$

$$i = 1, \ldots \# \text{ coordinates}.$$

Notice that the equations of motion do not define the lagrangian completely. Firstly, any constant multiple of a lagrangian produces the same equations of motion. It is important to fix this constant, because:

The lagrangian for a system comprising subsystems is the sum of the lagrangians of the subsystems.

For this summation to work, every subsystem must have the same multiplicative constant (more later). The standard convention is to use a scaling such that the coefficient of the scalar potential $V(q, t)$ is -1. This scaling also makes the hamiltonian, derived from the lagrangian, equal to the total energy under a few mild conditions.

Secondly, the total time derivative of any function can be added to the lagrangian, and produces the same equations of motion. Conventions are looser here, but all common physics has a well-defined agreed-upon lagrangian.

Note that we often use the typical lagrangian form of $L(q, \dot{q}, t)$ as representative of most, but not all, physical problems. If our problem has a lagrangian of this form, then our variations of q (and thus q-dot) are all the possible variations to make. Hamilton's principle of stationary action then leads to the typical Euler-Lagrange equations of motion.

However, some lagrangians include higher derivatives. [Lan] page 59b (bottom) gives a problem for finding the differential equation resulting from a lagrangian including 2nd derivatives. Pages 70+ gives an example of a physical elastic system (continuous, though) with $U(q(x), q'(x), q''(x))$ (potential energy) depending on the 2nd derivative $q''(x)$.

What Is A Derivative With Respect To A Derivative?

There is a notational shortcut that is universal in mechanics: that of defining the lagrangian as a function of q, $\dot{q}$, and t, where $\dot{q}$ is the time derivative of q. Then, we take partial derivatives with respect to $\dot{q}$! There's no new calculus here; it's just a shortcut to simplify the notation (though somewhat confuse the math).

To see this, recall that the lagrangian is a function of 3 variables; let's call them q, b, and t. Then

$$L = L(q, b, t) \qquad \Rightarrow \qquad \frac{\partial L}{\partial q}, \frac{\partial L}{\partial b}, \textit{and } \frac{\partial L}{\partial t} \text{ are all well-defined.}$$

But it happens that when evaluating the lagrangian $L(q, b, t)$, we will always plug-in $\dot{q}$ for b. So we don't bother introducing the variable b, and just write $\dot{q}$ everywhere. So where $\dot{q}$ appears in the lagrangian, it is just an argument of the lagrangian function. However, when considering the action integral over some path $q(t)$ (actual or variational), the lagrangian reduces to a function of time:

$$L = L(q(t), \dot{q}(t), t) \equiv L(t), \quad \text{which allows us to integrate} \qquad \textit{action} \equiv S = \int_{t_1}^{t_2} dt\, L(q(t), \dot{q}(t), t)$$

Hamilton's Principle: A Motivated Derivation of From F = ma

There is a long-standing, but rarely answered, question: why is Hamilton's Principle of stationary action true? Or similarly, how did Hamilton come up with it? Given the context of Hamilton's time (c. 1834), one can imagine a plausible train of thought leading to Hamilton's Principle. We motivate and *derive* here Hamilton's principle, starting with a very simple form. Next, we motivate the extension for magnetic forces. We then describe how the Principle rapidly extends to a broad class of important dynamics problems, including those with constraints of various forms. This wide application is reason

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enough to drive the exploration of Hamilton's Principle that ensued for over a hundred years, so that today, virtually all of fundamental physics can be given in Lagrangian form. The following derivation also introduces the power of coordinate-free methods, which are very important in many fields, such as tensor calculus, differential geometry, and Relativity. [I don't yet have the derivation of time-dependent lagrangians.??]

There are two important features of Hamilton's era: First, outside of classical mechanics, the mathematical theory of the calculus of variations was already well-known. From the brachistochrone problem, and others like it, the Euler-Lagrange (E-L) equations were known to make stationary an integral, with fixed endpoints, of the form:

$$\int_{x_1}^{x_2} f(y(x), y'(x))\, dx \qquad \textit{where} \quad x_1, x_2, y(x_1), \text{ and } y(x_2) \text{ are fixed .} \tag{5.1}$$

The function, $y(x)$, which makes the integral stationary with respect to small variations in $y(x)$, satisfies the E-L equation:

$$\frac{\partial}{\partial y} f(y, y') - \frac{d}{dx}\frac{\partial}{\partial y'} f(y, y') = 0 . \tag{5.2}$$

This is a differential equation in $y(x)$. Therefore, solving this E-L equation finds the $y(x)$ that makes (5.1) stationary.

Another feature of Hamilton's day was that there was a long-standing belief that nature was elegant and parsimonious, i.e. that the laws of nature satisfied minimization properties. For example, as early as c. 60 CE, Hero of Alexandria (Heron) stated a form of Fermat's principle of least time for light propagation. In keeping with this spirit, one might well ask: Do the laws of motion (classical mechanics) satisfy some minimization principle? Starting directly with $F = m\ddot{q} = \frac{d}{dt} p(\dot{q})$ in one dimension, we write:

$$F(q) - \frac{d}{dt} p(\dot{q}) = 0 \qquad \textit{where} \quad q \text{ is a cartesian coordinate}$$

$$F(q) \text{ is derivable from a scalar potential .}$$

We recognize this as having the form of the E-L equation (5.2), given the change in notation: $y \to q$, $x \to t$. Therefore, simple dynamics *do*, in fact, minimize some integral. We now find that integral.

In the new notation, the E-L equation (5.2) becomes:

$$\frac{\partial}{\partial q} f(q, \dot{q}) - \frac{d}{dt}\frac{\partial}{\partial \dot{q}} f(q, \dot{q}) = 0 \qquad \textit{where} \quad q \equiv q(t),\ \dot{q}(t) = \frac{d}{dt} q(t) . \tag{5.3}$$

We make this equivalent to $F(q) - \frac{d}{dt} p(\dot{q}) = 0$ by equating corresponding terms:

$$F(q) = \frac{\partial}{\partial q} f(q, \dot{q}), \qquad \frac{d}{dt} p(\dot{q}) = \frac{d}{dt}\frac{\partial}{\partial \dot{q}} f(q, \dot{q}) \quad \text{or} \quad p(\dot{q}) = m\dot{q} = \frac{\partial}{\partial \dot{q}} f(q, \dot{q}) .$$

We find $f(q, \dot{q})$, the integrand of the integral that is made stationary against small variations $\delta q(t)$ and $\delta \dot{q}(t)$, by taking the anti-derivatives (indefinite integrals) of $F(q)\, dq$, and $m\dot{q}\, d\dot{q}$:

$$f(q, \dot{q}) = \int F(q)\, dq = -U(q) + \text{a function of } \dot{q},$$

$$f(q, \dot{q}) = \int m\dot{q}\, d\dot{q} = \frac{1}{2} m\dot{q}^2 = T(\dot{q}) + \text{a function of } q .$$

The constant of integration is irrelevant. Together, these imply:

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$$f(q,\dot{q}) = T(\dot{q}) - U(q) \qquad \text{and} \qquad \int_{t_1}^{t_2} f(q,\dot{q})\, dt \text{ is stationary .} \tag{5.4}$$

We rename *f*( ) to *L*( ), and call it the **lagrangian**. We define the integral as *S*, and call it the **action**.

This is a most satisfying result, for two reasons: first, *f*( ) is a scalar function of the dynamics, and therefore coordinate-free (more shortly). Second, *f*( ) is simply the difference between two well-known and physically meaningful quantities.

Since the integral is coordinate free, it immediately gives us generalized coordinates: if the integral is stationary over *any* small variations in the trajectory *q*(*t*), it is stationary over small variations in *all* coordinates, even oblique (non-orthogonal) ones. Since stationarity implies the E-L equation [see any mechanics book ever], the E-L equation is valid in arbitrary coordinates.

We note briefly that any potential energy allows for an arbitrary additive constant; therefore, so does the lagrangian. This is consistent with the E-L equation containing only derivatives of the lagrangian, so the E-L equation is insensitive to any additive constant in the lagrangian.

We have derived Hamilton's Principle in its simplest form. We now show that it easily extends to a much wider range of applications. An obvious extension is: what about 2D or 3D motion? In 3D, starting again with cartesian coordinates, the equations of motion separate:

$$q_i \equiv \{q_x, q_y, q_z\}: \;\; F_x(q_i) - \frac{d}{dt} p_x(t) = 0, \qquad F_y(q_i) - \frac{d}{dt} p_y(t) = 0, \qquad F_z(q_i) - \frac{d}{dt} p_z(t) = 0 .$$

This leads directly to the straightforward extension of the integrand (5.4). Since the 3 coordinates have independent motions, minimizing each coordinate's action individually is equivalent to minimizing their sum. Therefore, we extend the definition of *L* to be the sum of the 3 lagrangians from the 3 coordinates. The action is similarly extended. Skipping some details, this gives:

$$L\big(q_i(t), \dot{q}_i(t)\big) \equiv \left(\underbrace{\sum_{i=1}^{\#\,coordinates} \frac{1}{2} m \dot{q}_i^{\,2}}_{T(\dot{q}_i)} \right) - U(q_i), \qquad S \equiv \int_{t_1}^{t_2} L\big(q_i(t), \dot{q}_i(t)\big)\, dt .$$

Again, making the action stationary over small variations in 3D immediately implies the validity of generalized, oblique 3D coordinates.

We should note that although Hamilton's principle is valid in 3D, evaluating the kinetic energy *T* in oblique coordinates is often difficult, because v^2 requires taking a dot-product (or at least, a vector magnitude). A similar problem appears later when we include magnetics.

Another obvious extension is multiple particles. Again, each can be treated separately at first, and again, minimizing each particle's individual action is equivalent to minimizing the total action of the system. Skipping some details, our above definitions of "lagrangian" and "action" are now valid even when the generalized coordinates span multiple particles.

Applying what we've learned to the ubiquitous simple harmonic oscillator (SHO), we find that the action is not necessarily minimized, because the action of SHO motion is not, in fact, a minimum with respect to variations in *q*(*t*), but it *is* stationary under such variations. This is fine, because the E-L equations we started with never guaranteed minima in the first place, only that the action integral is stationary.

What about time-varying potentials? It turns out, the existing formulas work just as well with them. [I'm not sure how to do this yet?? Perhaps, we apply our current E-L equations over the infinitesimal interval t_1 to $t_2 \equiv t_1 + dt$:

$$dS \equiv \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt .$$

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Over this interval, the explicit time dependence of L can be ignored, and our existing E-L equations are valid. We'd like to stitch together a large number of such intervals to create a finite interval, but our current E-L equations only apply when the endpoints at t_1 and t_2 are fixed. An arbitrary variation $\delta q(t)$ over a finite interval would not be fixed at the endpoints of each infinitesimal interval. I'll bet there's a way to fix this, but it's not obvious to me.]

What about magnetic forces? The Lagrangian formalism we have developed so far is so powerful, that we are highly motivated to see if we can push it to cover even more physical situations. Is there a lagrangian that will produce the Lorentz force: $F = e\mathbf{v} \times \mathbf{B}$? While there is no direct derivation of such a lagrangian, [Jose and Saletan, 1998] point out that, if it exists, it must be linear in both $\mathbf{v}$ and $\mathbf{B}$, and proportional to the charge e. It must also be a scalar, and some kind of "potential," so the ansatz $e\mathbf{v}\cdot\mathbf{A}$ is the minimal lagrangian meeting these criteria. Direct substitution reveals that this term does, in fact, yield the proper equation of motion (in SI units). The resulting classical lagrangian, covering many particles in many dimensions (i.e., many degrees of freedom), and magnetism is:

$$L(q_i, \dot{q}_i) \equiv T(\dot{q}_i) - U(q_i) + \sum_{j=1}^{\#\text{ particles}} e_j \mathbf{v}_j \bullet \mathbf{A}(\mathbf{r}_j) \,.$$

Note that both the velocities $\mathbf{v}_j$ and the dot-products in the last term may be difficult to evaluate in oblique coordinates.

We also note that since $\mathbf{A}$ has gauge freedom, so does the lagrangian. In fact, our lagrangian already has another "gauge freedom" from early on: adding to the lagrangian the total time derivative of any function of the coordinates $\{q_i(t)\}$ and time has no effect on the resulting E-L equations of motion. This is because such a function adds a fixed constant to the action, regardless of the trajectory. Such a constant has no effect on which trajectories make the action stationary, and so doesn't affect the equations of motion. (The gauge function can depend on $q(t)$ and t, but not the coordinate velocities $\dot{q}(t)$, because they are *not* fixed at the endpoints of the action integral.)

Note that a development similar to magnetics allows the inclusion of the Coriolis acceleration when working in a rotating frame of reference. The centrifugal force is also accommodated by a simple, scalar potential [*Funky Mechanics Concepts*].

Sometimes, it is advantageous to define additional generalized coordinates beyond what we need, especially if we must compute the forces required to constrain the motion in some way. Such constraints can be incorporated into Lagrangian mechanics by introducing Lagrange multiplier functions of time, $\lambda(t)$. Details are available in standard texts [*Funky Mechanics Concepts*].

Given the success of Lagrangian mechanics in the wide range of applications already described, physicists are eager to apply its methods to even more situations. For example, can Special Relativity be described with lagrangian mechanics? As with the magnetic force, one can constrain the possible forms of such a lagrangian, and a little trial and error reveals that SR *can* be accommodated. Note, however, that the relativistic lagrangian is *not* $T - U$, even when using the relativistic kinetic energy. (However, the relativistic hamiltonian *is* the relativistic total energy.] The magnetic lagrangian, just like the Lorentz force law, is already relativistically valid.

Another big discipline in mechanics is that of deformable continua. Continua have infinite degrees of freedom, because every point in the "solid" can move. Casting the system in lagrangian form requires defining generalized coordinates that describe the deformations as functions of both time and space, and incorporating a new derivative term, the spatial derivative (gradient). The "lagrangian" itself is then built up from a **lagrangian density** (essentially "lagrangian per unit volume"):

$$\mathcal{L} = \mathcal{L}(q(t,\mathbf{x}), \dot{q}(t,\mathbf{x}), \nabla q(t,\mathbf{x}), t), \qquad \text{and} \qquad L(q, \dot{q}, \nabla q, t) \equiv \int_{space} \mathcal{L}\, d^3\mathbf{r} \,.$$

We see that this form leads naturally to a relativistic scalar definition of the action:

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$$S \equiv \int_{t_1}^{t_2} L\, dt = \int_{t_1}^{t_2} \left(\int_{space} \mathcal{L}\, d^3\mathbf{r} \right) dt = \int_{spacetime} \mathcal{L}\, d^4\mathbf{x}\,.$$

With continua and relativistic scalar action now covered, quantum field theory (QFT) becomes susceptible to yet another generalization of Hamilton's Principle of stationary action. Furthermore, in a reach to one of the farthest corners of physics, it turns out that even the Einstein field equations of General Relativity can be written in lagrangian form.

Hamilton's Principle is so fundamental and ubiquitous across physics that when a new idea is formulated, physicists often look early-on for a lagrangian to describe it. Sometimes, journal reviewers explicitly request a "least-action" formulation [Magueijo, 2003]. Clearly, the principle of stationary action has grown far beyond anything that William Rowan Hamilton could have imagined back in 1834.

Hamilton's Principle of Stationary Action: A Variational Principle

We here derive the equations of motion from Hamilton's Principle, the reverse of what we did above. Define $\mathbf{q}(t)$ as the set of position functions: $\mathbf{q}(t) \equiv \left(q_1(t), \ldots q_n(t)\right)$. These may describe the actual trajectory of a system through configuration space, or they may be a hypothetical trajectory. Hamilton's principle defines the action, $S[\mathbf{q}(t)]$, as a functional of $\mathbf{q}(t)$, and states:

For trajectories near the actual trajectory of a system, the variation of the functional S is zero (i.e., the action is stationary to first order in small variations of the trajectory).

Given the positions of a system at some time, $\mathbf{q}_1(t_1)$, and its positions later, $\mathbf{q}_2(t_2)$, how can we find the trajectory $\mathbf{q}(t)$ of the system between the 2 points? Note that *velocities* are unknown, even at the endpoints t_1 and t_2; solving for $\mathbf{q}(t)$ also solves for the velocities everywhere. Hamilton's principle says the system trajectory between the two pairs of given points and times makes the **action** functional stationary between them:

$$\mathbf{q}(t) \text{ makes stationary} \qquad S[\mathbf{q}(t)] \equiv \int_{\mathbf{q}_1}^{\mathbf{q}_2} dt\, L(\mathbf{q}, \dot{\mathbf{q}}, t)\,.$$

(Often, $\mathbf{q}(t)$ makes the action minimum, but not necessarily.)

The stationary paths of the action functional $S[\mathbf{q}(t)]$ are found by the vanishing of its first order variation $\delta S[\mathbf{q}(t)]$ for arbitrary infinitesimal variations $\delta\mathbf{q}(t)$ of the path connecting ($\mathbf{q}_1$, t_1) to ($\mathbf{q}_2$, t_2). To compute $\delta S[\mathbf{q}(t)]$, substitute $\mathbf{q}(t) + \delta\mathbf{q}(t)$ for $\mathbf{q}(t)$ in the definition of action, expand to first order in $\delta\mathbf{q}(t)$, and integrate the 2nd term by parts (JH 3.38 p 44):

$$S[\mathbf{q}(t)] = \int_{t_1}^{t_2} dt\, L(\mathbf{q}, \dot{\mathbf{q}}, t), \qquad NB: \delta\dot{\mathbf{q}}(t) = \frac{d}{dt}\delta\mathbf{q}(t)$$

$$\delta S[\mathbf{q}(t)] = \int_{t_1}^{t_2} dt\, \delta L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial \mathbf{q}(t)} \delta\mathbf{q}(t) + \frac{\partial L}{\partial \dot{\mathbf{q}}(t)} \delta\dot{\mathbf{q}}(t) \right)$$

$$where \quad \frac{\partial}{\partial \mathbf{q}} \equiv \nabla_{\mathbf{q}} = \left(\frac{\partial}{\partial q_1}, \ldots \frac{\partial}{\partial q_n} \right)$$

Now we integrate by parts to eliminate $\delta\mathbf{q}$-dot in favor of $\delta\mathbf{q}$, noting that $\delta\mathbf{q}(t_1) = \delta\mathbf{q}(t_2) = \mathbf{0}_v$:

$$IBP: \quad U\, dV \;\rightarrow\; -V\, dU$$

$$Let \quad U = \frac{\partial L}{\partial \dot{\mathbf{q}}(t)}, \qquad dV = \delta\dot{\mathbf{q}}(t)\, dt \quad \Rightarrow \quad dU = \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}(t)} dt, \qquad V = \delta\mathbf{q}(t)$$

$$\delta S[\mathbf{q}(t)] = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial \mathbf{q}(t)} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}(t)} \right) \delta\mathbf{q}(t) + \cancel{\frac{\partial L}{\partial \dot{\mathbf{q}}(t)} \delta\mathbf{q}(t) \Big|_{t_1}^{t_2}} \quad \Rightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}(t)} - \frac{\partial L}{\partial \mathbf{q}(t)} = 0$$

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This can be thought of as a generalized Newton's 2nd law: "rate of change of momentum = sum of forces." This equation is **Lagrange's Equation of Motion** (LEM), or just Lagrange's Equation.

Note that since the small variations of the trajectory are *arbitrary*, the $\mathbf{q}(t)$ can be *arbitrary* generalized coordinates, so long as they fully specify the positions over time of all degrees of freedom of the system. In other words:

Hamilton's principle implies that for *any* generalized coordinates,
Lagrange's equations solve for the motion.

Hamilton's Principle: Why Isn't "Stationary Action" an Oxymoron?

What is the significance of "least action" vs "stationary action"?

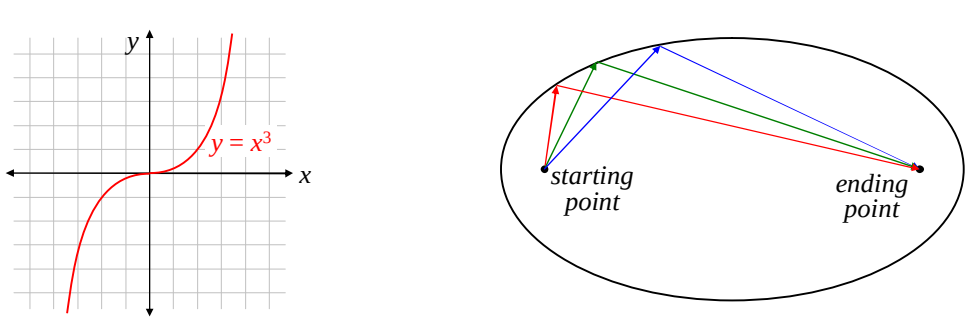
The action for the motion of a system is stationary: either minimum, or non-decreasing.

Action is *not* necessarily minimized, and it is *never* maximized [Ref??]. A stationary action might be neither a minimum nor a maximum. All minima and maxima are stationary paths, but not all stationary paths are minima or maxima. [Note that some authors use the term "extremize" nonstandardly to mean "make stationary." Similarly, they may use the term "extremum" nonstandardly to mean "stationary path."]

To illustrate, let's recall a simpler case: a stationary point of a function: In calculus, a function can have a zero derivative in 3 cases: a maximum, a minimum, or an inflection point. Zero derivative is where the function is "stationary": there is no (first-order) change in the function for tiny changes in its argument, i.e.

$$\textit{when} \quad f'(x) = 0: \qquad f(x+dx) = f(x) + O(dx^2)\,.$$

E.g., $f(x) = x^3$ has zero derivative at $x = 0$, but that is neither a maximum nor a minimum. It is an inflection point.



(Left) A stationary point of $y = x^3$. (Right) Reflections from the inside of an ellipse follow paths of stationary action, but are neither minimum nor maximum.

Thus, a stationary point doesn't have to be a minimum; it could be a maximum, or even just a "flat spot" in the function, which is neither a maximum nor a minimum.

So it is with a path whose action is stationary: it might be the least of any nearby actions, or it could be just a big flat spot among many paths.

As an example closer to stationary action, the diagram above right shows the case of stationary action for lots of adjacent paths: reflections from inside an ellipse. Imagine a shiny ellipse. If you shine a flashlight at the wall, the light will reflect and go through the other focus. So the path from one focus to the other is one of stationary action. But shining the flashlight in *any* direction will send light through the other focus in exactly the same amount of time. No one path is preferred over the other, and *all* the actions of all the directions are the same. The action on any path is stationary, but it is neither larger nor smaller than nearby paths. It is exactly equal to other paths that go in straight lines, bounce off the wall, and through the other focus.

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The original Hamilton's principle is a little weird because it is not predictive: you have to know the position and time of the starting point, and the position and time of the endpoint. Hamilton's principle then tells you *how* the particle got from "here" to "there" (i.e., what path it took). In this form, though, it doesn't tell you where the particle will be in the future. However, by analysis of Hamilton's principle, we *can* find the equations of motion which do predict the future. Euler and Lagrange did this analysis, and their equations are called the **Euler-Lagrange equations of motion**. Of course, they are the same as Newton's equations of motion for those cases that Newton studied, but the E-L equations of motion are more general: they work for any physics describable by an action (which is all of known microscopic physics).

TBS: Examples of non-minimum action: harmonic oscillator (local minimum), multiple reflections between plane mirrors (stationary).

Addition of Lagrangians

Consider a system comprising multiple parts, each part of which has its own individual lagrangian. Amazingly, the lagrangian for the whole system is simply the *sum* of the individual lagrangians. When dealing with individual lagrangians and the principle of stationary action, the lagrangians are only defined to within a multiplicative constant, i.e. multiplying a lagrangian by any constant produces an equivalent lagrangian, that produces the same physics. However, when adding two such lagrangians to produce the total lagrangian for the aggregate system, the multiplicative constants between the two lagrangians matters. Both must be on the same "scale," so to speak. After they are added, then any multiplicative constant on the aggregate lagrangian is again arbitrary.

The scale factor for individual lagrangians are well defined by the requirement that they produce the correct equations of motion when combined with each other. Physicists pretty much all agree on the scale factors for lagrangians, and lagrangians in text books are always written with the proper scale factor.

A trivial example is a system of 2 nonrelativistic particles:

$$L_1 = T_1 - V_1, \qquad L_2 = T_2 - V_2 \qquad \Rightarrow \qquad L_{total} = L_1 + L_2 = T_1 - V_1 + T_2 - V_2 = T_{total} - V_{total}$$

Velocity Dependent Forces

There are two very common velocity-dependent forces: the magnetic Lorentz force, $\mathbf{F} = \mathbf{v} \times \mathbf{B}$, and the Coriolis force, $\mathbf{F} = -2\boldsymbol{\omega} \times \mathbf{v}$. Both of these act perpendicular to the velocity, and therefore do no work (energy is unchanged), and produce circular orbits.

This section describes how the cross-product velocity terms can be written in-plane as derivatives of the "vector potential" [in fact, as the **exterior derivative** of the vector potential. The exterior derivative essentially reduces to the curl in 3 dimensions.]

Just as rotations, in general, are not about an axis, but rather *in a plane*,
the magnetic field can be thought of not as a vector pointing in some direction,
but as a potential field acting in a plane.

This planar field can be broken up into 3 components, the *x*-*y* component, the *y*-*z* component, and the *z*-*x* component. The *x*-*y* component corresponds to B_z, the *y*-*z* component to B_x, and the *z*-*x* component to B_y.

Let us consider the *x*-*y* (or equivalently, the B_z) component of the magnetic field. It is derivable from the magnetic vector potential $\mathbf{A}(x, y, z)$ as:

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}, \qquad F_x = v_y B_z = v_y \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right), \qquad F_y = -v_x B_z = -v_x \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

Note that each component of $\mathbf{A}(x, y, z)$, i.e., $A_x(x, y, z)$, $A_y(x, y, z)$, and $A_z(x, y, z)$, is a function of all 3 space coordinates, so A_x changes when we move in the *y* direction.

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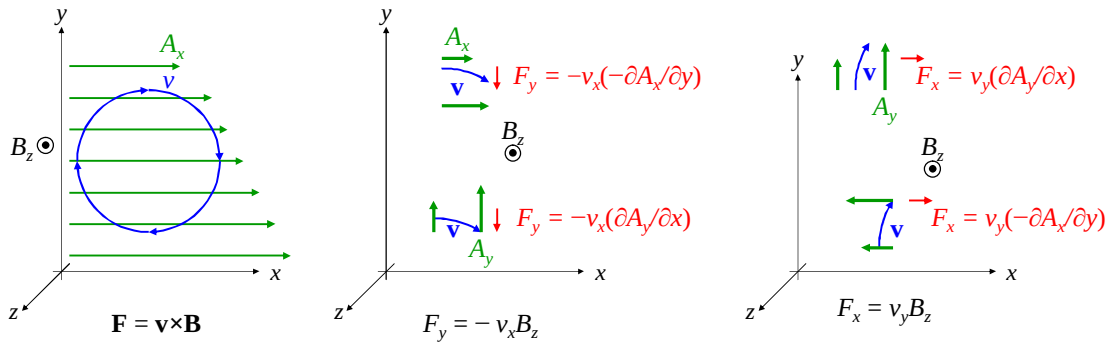

Velocity-dependent forces change the basic lagrangian from $T - V$ to (defining $\mathbf{q} \equiv (q_1, q_2, q_3) = (x, y, z)$)

$$L = T(\dot{\mathbf{q}}) - V(\mathbf{q}) + k\dot{\mathbf{q}} \cdot \mathbf{A}(\mathbf{q}) \text{ where } \quad k \equiv \text{constant giving the strength of the force .}$$

It is remarkable that such a simple term in the lagrangian, $k\mathbf{q}\cdot\mathbf{A}$, a scalar, produces the much more involved equations of motion. Let's see how this works for a charged particle in a static magnetic field (gaussian units):

$$\text{Lagrangian}: \quad L = T + \frac{e}{c}\dot{\mathbf{q}} \cdot \mathbf{A} = \sum_{j=1}^{dof} \left(\frac{1}{2} m \dot{q}_j^{\,2} + \frac{e}{c} \dot{q}_j A_j(\mathbf{q}) \right), \qquad \text{where} \quad e \equiv \text{charge of the particle}$$

$$\text{Lagrange's Eq}: \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i} \quad \rightarrow \quad \frac{d}{dt}\left(p_i(t) + \frac{e}{c} A_i(\mathbf{q}) \right) = \frac{e}{c} \sum_{j=1}^{dof} \dot{q}_j(t) \frac{\partial A_j(\mathbf{q})}{\partial q_i}$$

$$\text{where} \quad dof \equiv \text{mechanical degrees of freedom, i.e. \# of generalized coordinates}$$

[Note that it is not convenient to write these equations of motion in vector form, because the right hand side is awkward in vector notation.] These are the equations of *motion*, so $d\mathbf{A}/dt$ is a result of the particle moving in space, not because $\mathbf{A}(\mathbf{q})$ is changing in time:

$$\text{Define} \qquad \mathbf{A}(t) \equiv \mathbf{A}\big(\mathbf{q}(t)\big) \quad \Rightarrow \quad A_i(t) \equiv A_i\big(\mathbf{q}(t)\big)$$

$$\text{Then} \qquad \frac{dA_i(t)}{dt} = \sum_{j=1}^{dof} \frac{dq_j(t)}{dt} \frac{\partial A_i(\mathbf{q})}{\partial q_j} = \sum_{j=1}^{dof} \dot{q}_j(t) \frac{\partial A_i(\mathbf{q})}{\partial q_j}$$

$$\text{Eqs of motion}: \quad \frac{dp_i}{dt} + \frac{e}{c} \sum_{j=1}^{dof} \dot{q}_j(t) \frac{\partial A_i(\mathbf{q})}{\partial q_j} = \frac{e}{c} \sum_{j=1}^{dof} \dot{q}_j(t) \frac{\partial A_j(\mathbf{q})}{\partial q_i}$$

$$\frac{dp_i}{dt} = \frac{e}{c} \sum_{j=1}^{dof} \dot{q}_j(t) \left(\frac{\partial A_j(\mathbf{q})}{\partial q_i} - \frac{\partial A_i(\mathbf{q})}{\partial q_j} \right) \qquad NB: \left(\frac{\partial A_j(\mathbf{q})}{\partial q_i} - \frac{\partial A_i(\mathbf{q})}{\partial q_j} \right) \equiv \nabla \times \mathbf{A}$$

In words, $\partial L / \partial \dot{q}$ on the left removes $\dot{q}$ and leaves $\mathbf{A}(\mathbf{q})$. Then the d/dt turns $\mathbf{A}(\mathbf{q})$ into $\dot{\mathbf{q}} \cdot \nabla \mathbf{A}$. On the right, the $\partial L/\partial q$ leave the $\dot{q}$ alone, and acts on $\mathbf{A}(\mathbf{q})$ to also produce $\dot{q}\ \partial\mathbf{A}/\partial q_i$ terms. These terms from the left and right combine to produce the "curl" of $\mathbf{A}$.

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The last term cancels the $j = i$ term from the sum, so that force in the *i direction* involves velocities in *all directions except i*. But from the point of view of *planes* instead of *directions*, we say that the Lorentz force in the *x-y plane* is completely determined by the *x-y* components of the vector potential **A**. This is the true nature of the magnetic field:

The magnetic forces in a plane are completely determined by the vector potential components in that plane.

In traditional terms, the *x-y* components of **A** completely determine B_z, but then (for a given velocity in the *x-y* plane) the Lorentz force $\mathbf{F} = \frac{e}{c}\mathbf{v}\times\mathbf{B}$ in the *x-y* plane is determined completely by B_z, which is to say, the Lorentz force in the *x-y* plane is determined completely by the *x-y* components of **A**. In other words, potentials in the plane, stay in the plane.

We've now broken the magnetic forces into planes, instead of vectors. But we must remember that each direction in space is a member of *two* planes: *x* is included in the *x-y* plane and the *z-x* plane, and similarly for *y* and *z*. Therefore, the *x*-component of force is the *sum* of the forces from the *x-y* plane and the *z-x* plane. This is why the force in the *x* direction involves velocities in both the *y* and *z* directions.

Coriolis forces can similarly be taken account by introducing a Coriolis vector potential. See *Funky Electromagnetic Concepts* for discussions of finding the vector potential for a given magnetic field (or other velocity-dependent force). You can use the analogy

$$F_{Coriolis} = 2m\mathbf{v}\times\boldsymbol{\omega} \quad \leftrightarrow \quad F_{Lorentz} = \frac{e}{c}\mathbf{v}\times\mathbf{B}, \qquad so \qquad \underbrace{m \leftrightarrow \frac{e}{c}}_{\substack{properties\ of \\ the\ particle}}, \ and \ \ 2\boldsymbol{\omega} \leftrightarrow \mathbf{B}.$$

Non-potential Forces

Some forces (e.g., static or kinetic friction, viscous drag) cannot be derived from a potential function (they are "polygenic forces" in Lanczos' terminology). Such forces can be incorporated into Lagrange's equations by simply writing them on the RHS [Gol p23]:

$$\frac{\partial L}{\dot{q}_i} - \frac{d}{dt}\left(\frac{\partial L}{q_i}\right) = Q_i(t), \qquad where \quad Q_i(t) \text{ are the non-potential-derivable forces .}$$

Examples TBS: kinetic friction, viscous friction. Friction work function of [F&W].

Lagrangians for Relativistic Mechanics

At relativistic speeds, the lagrangian is *not* $T - V$ (see *Funky Relativity Concepts* for more information). The relativistic lagrangian for a charged particle is (in Gaussian units):

$$L(\mathbf{q},\mathbf{v}) = -mc^2\sqrt{1 - v^2/c^2} + \frac{e}{c}\mathbf{v}\cdot\mathbf{A} - V(\mathbf{q}) \qquad \text{[LL2 16.4 p48]}$$

$$where \quad e \equiv \text{charge of particle}, \qquad V(\mathbf{q}) \equiv \text{potential of particle} = e\phi(\mathbf{q}) \text{ for an electric field}$$

Fully Functional

The concept of functionals is too often glossed over so quickly that it makes no sense. With a few simple definitions and examples, functionals and functional derivatives are readily understood. This provides a solid foundation for their use in continuum materials (strings, magnets, fluids, etc.), and in theoretical analysis, such as classical mechanics. This section requires only simple calculus. We follow this course:

- Definition of a functional, with examples.
- Functional derivatives, with examples.

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- Superposition of small variations, i.e. linearity of functional derivatives.
- A slightly different view of functionals as functions of an infinite number of arguments.
- Sample application: classical mechanics and Euler-Lagrange equations.

A **functional** takes a function as input and produces a number from it. [Contrast with a *function*, which takes a number (or a set of them) and produces a number.] A simple functional might be:

$$W[f(x)] = \int_0^1 f(x)\,dx\,.$$

Square brackets around the function-argument is standard notation for a functional. One of the best known functionals in physics is the **action functional**, $S[\ ]$; it acts on the lagrangian, and produces the action, S (a number):

$$S\left[L(t)\right] \equiv \int_{t_1}^{t_2} L(t)\,dt\,.$$

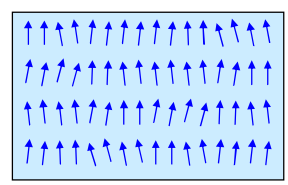
However, the lagrangian is a function of the coordinates and velocities, $q(t)$ and $\dot{q}(t)$, which are themselves functions of time. Therefore, the action can be considered a functional of the coordinates and their velocities:

$$S\left[q(t),\dot{q}(t)\right] \equiv \int_{t_1}^{t_2} L\left(q(t),\dot{q}(t),t\right)dt\,.$$

Even though $q(t)$ and $\dot{q}(t)$ are related, they are treated as independent variables, because there is so much flexibility in their values that they are, for most practical purposes, independent.

We'll return to this point when we discuss functional derivatives.

Another example of a functional is the hamiltonian of a continuous medium, such as a string or the magnetization of a material.



(Left) A vibrating, stretched string has kinetic and potential energy.
(Right) Magnetic fields and misalignment of magnetization both store potential energy.

For a stretched string (above left), the hamiltonian is

$$H\left[y(x)\right] = \int_0^L dx\left[\frac{\mu}{2}\dot{y}^2(x) + \frac{\tau}{2}\left(y'(x)\right)^2\right] \qquad \textit{where} \quad \begin{cases} \mu \text{ is the mass density (per unit length)} \\ \tau \text{ is the string tension} \\ y(x) \equiv \text{string displacement at position } x \end{cases}$$

Notice that in this case, the functional "took derivatives" of its argument, $y(x)$. This is a common shorthand: the definition of a functional may include differential operators. A more complete notation would write the hamiltonian explicitly as a function of $y(x), \dot{y}(x)$, and $y'(x)$: $H\left[y(x),\dot{y}(x),y'(x)\right] = \ldots$, as we did for the action above, a functional of $q(t)$ and $\dot{q}(t)$.

For a magnetized material (above right), the hamiltonian might look like:

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$$H\left[m(\mathbf{x})\right] = \int d^3x \left[\frac{k}{2}m^2(\mathbf{x}) + t\left(\nabla m(\mathbf{x})\right)^2\right] \qquad \textit{where} \quad \begin{cases} \mathbf{x} \text{ is the position vector in the body} \\ m(\mathbf{x}) \equiv \text{magnetization at position } \mathbf{x} \\ k,\, t \text{ are constants} \end{cases}$$

Here again, the functional "took the gradient" of its argument, $m(\mathbf{x})$. In this new shorthand, we could omit the $\dot{q}(t)$ from the action functional, where it is understood that time derivatives of the functional argument $q(t)$ may be used in the functional:

$$S\left[q(t)\right] \equiv \int_{t_1}^{t_2} L\left(q(t), \dot{q}(t), t\right) dt \qquad \text{(shorthand functional arguments)}\ .$$

If we make a small change in our function, call it $\delta q(t)$, we will get a small change in the action, δS:

$$\delta S\left[q(t), \dot{q}(t), \delta q(t), \delta\dot{q}(t)\right] \equiv S\left[q(t) + \delta q(t),\, \dot{q}(t) + \delta\dot{q}(t)\right] - S\left[q(t),\, \dot{q}(t)\right]$$

Note that the change in S depends not only on the small change $\delta q(t)$, but also on the function $q(t)$ which we are deviating from. This makes δS a functional of 4 functions, as above.

Functional derivatives: Just as a *function* derivative describes the response of a function to small changes in its argument, a **functional derivative** describes the response of the functional to small changes in its function argument. We write functional derivates with "δ". For example:

$$\text{Given:} \quad W[f(x)], \quad \frac{\delta W}{\delta f} \equiv K(x) \quad \textit{such that} \quad \delta W = \int K(x)\delta f(x)\, dx\ .$$

Since we seek δW (a number) given $\delta f(x)$ (a function), you might think that the functional derivative is also a functional, but it's not.

A functional derivative is a *function*: it is the kernel function for integrating $\delta f(x)$ into δW.

Note that the operation of integration with a kernel *is* indeed a functional; in this case, it acts on the function $\delta f(x)$, to produce a number, δW. A key aspect of integration with a kernel is that it is a *linear* operation on the variation of the function argument, $\delta f(x)$. This means superposition applies (discussed more shortly).

Some functionals do not have functional derivatives, for example, a functional which chooses the largest value of its function argument. The change in the functional cannot be written as an integral operation on the change in function argument, therefore the functional derivative is not defined.

A functional of two arguments has two partial functional derivatives: one with respect to the first argument, and another with respect to the 2nd argument. This is analogous to a function of two arguments which has two partial derivatives. For example,

$$\textit{Given } H[m(\mathbf{x}), \nabla m(\mathbf{x})] = \int d^3x \left[\frac{k}{2}m^2(\mathbf{x}) + t\left(\nabla m(\mathbf{x})\right)^2\right],$$

$$\textit{then} \qquad \frac{\delta H}{\delta m} \equiv K_1(\mathbf{x}), \qquad \frac{\delta H}{\delta \nabla m} \equiv K_2(\mathbf{x})$$

$$\Rightarrow \qquad \delta H = \int d^3x \left[K_1(\mathbf{x})\delta m(\mathbf{x}) + K_2(\mathbf{x})\delta\nabla m(\mathbf{x})\right]$$

$$= \int d^3x \left[km\,\delta m + 2t\left(\nabla m\right)\delta\nabla m\right]$$

The functional derivative is only valid in the limit that variations in *all* its function argument(s) are infinitesimal everywhere in the domain of interest. For example, the functional derivative of the action S is valid only when both $\delta q(t)$ is everywhere small *and* also $\delta\dot{q}(t)$ is everywhere small, over the time interval $[t_1,\ t_2]$. Note that δq being small does not guarantee that $\delta\dot{q}(t)$ is small; consider a series of small,

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instantaneous steps in δq (below left). δq is small, but the velocity $\delta\dot{q}(t)$ is infinite at the steps. Conversely, $\delta\dot{q}(t)$ being small does not insure that δq is small; consider a large δq which varies slowly (below right).

(Left) $\delta q(t)$ is small, but $\delta\dot{q}(t)$ is not. (Right) $\delta\dot{q}(t)$ is small, but $\delta q(t)$ is not.

It is in this sense that the functional and functional derivative treat $\delta q(t)$ and $\delta\dot{q}(t)$ as independent. Similarly, if $H[m(\mathbf{x})]$ depends on $\nabla m(\mathbf{x})$, then $\delta H/\delta m$ is meaningful only for both $\delta m(\mathbf{x}) \to 0$ for all $\mathbf{x}$, *and* $\delta\nabla m(\mathbf{x}) \to 0$ for all $\mathbf{x}$.

In physics, we often look for minima (or stationary paths) of functionals, such as minimum energy, or minimum action. This means the functional derivative with respect to *all* arguments (explicit, *and* implicit from derivative operators in the functional) are zero. Then we might write, in the shorthand notation,

$$\textit{Given } \delta S\left[q(t), \delta q(t)\right] \textit{ is stationary}, \qquad \Rightarrow \qquad \frac{\delta S}{\delta q} = 0, \quad \textit{AND} \quad \frac{\delta S}{\delta \dot{q}} = 0\,.$$

But again, $\delta q(t) \to 0$ does not imply $\delta\dot{q}(t) \to 0$. In mechanics, references often find the path of stationary action without writing it explicitly in the form of a functional derivative. However, we *are* finding the functional derivative. Since $\delta S = 0$ for arbitrary $\delta q(t)$, this requires the functional derivative be identically 0. Note also that in deriving the Euler-Lagrange equations, we first take a functional derivative, which acts as if $\delta q(t)$ and $\delta\dot{q}(t)$ are independent. However, to complete the derivation, we crucially must make use of the complete dependence of $\delta\dot{q}(t)$ on $\delta q(t)$ (see any mechanics text).

Superposition: In *function* derivatives, we have a principle of superposition for small changes in the argument, i.e. given two small changes, the result of both changes together is the sum of the individual changes.

$$\textit{Given } f(a), da_1 \textit{ and } da_2 \quad \textit{then} \qquad df \equiv f(a + da_1 + da_2) - f(a) = \frac{df}{da}\left(da_1 + da_2\right) = \frac{df}{da}da_1 + \frac{df}{da}da_2.$$

Also, for functions of two or more arguments:

$$\textit{Given } f(a,b), da, \textit{ and } db \quad \textit{then} \qquad df \equiv f(a + da, b + db) - f(a,b) = \frac{\partial f}{\partial a}da + \frac{\partial f}{\partial b}db$$

Similarly, with functional derivatives, we have a principle of superposition:

$$\delta S\left[\delta q_1(t) + \delta q_2(t)\right] = \delta S\left[\delta q_1(t)\right] + \delta S\left[\delta q_2(t)\right]$$

This follows directly from the equation for δS in terms of the functional derivative, $K(t)$:

$$\begin{aligned}\delta S[\delta q_1(t) + \delta q_2(t)] &= \int_{t_1}^{t_2} dt\; K(t)\left(\delta q_1(t) + \delta q_2(t)\right) = \int_{t_1}^{t_2} dt\; K(t)\delta q_1(t) + \int_{t_1}^{t_2} dt\; K(t)\delta q_2(t) \\ &= \delta S[\delta q_1(t)] + \delta S[\delta q_2(t)]\end{aligned}$$

It is this requirement for linearity (aka superposition) in the variation of a functional that insures that all functional derivatives can be written as kernel functions, which can be integrated with the argument variation to produce the functional variation. Integration with a kernel function is *the most general* linear operation that can be performed on another function. It is analogous to taking the dot product of a finite-dimensional vector with some constant vector: taking a dot product with a constant vector is the most

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general linear operation that can be performed on another vector. The continuum limit of infinite dimension takes the dot product into integration with a kernel function.

Given the lagrangian for the action S, we can evaluate the functional derivative of the action in terms of the lagrangian:

$$\delta S[\delta q(t),\delta\dot{q}(t)] = \int_{t_1}^{t_2} dt\left[L\left(q(t)+\delta q(t),\dot{q}(t)+\delta\dot{q}(t)\right) - L\left(q(t),\dot{q}(t)\right)\right]$$

$$= \int_{t_1}^{t_2} dt\left[\frac{\partial L}{\partial q}\delta q(t) + \frac{\partial L}{\partial \dot{q}}\delta\dot{q}(t)\right] = \int_{t_1}^{t_2} dt\left[K_1\delta q(t) + K_2\delta\dot{q}(t)\right]$$

$$\Rightarrow \qquad \frac{\delta S}{\delta q} \equiv K_1(q(t),\dot{q}(t)) = \left.\frac{\partial L}{\partial q}\right|_{q(t),\dot{q}(t)}$$

$$\textit{and} \qquad \frac{\delta S}{\delta \dot{q}} \equiv K_2(q(t),\dot{q}(t)) = \left.\frac{\partial L}{\partial \dot{q}}\right|_{q(t),\dot{q}(t)}$$

We have written the functional derivative of S in terms of partial derivatives of L. As previously mentioned, the functional derivative is evaluated along a given path, much like a function derivative is evaluated at a given point. As a simpler example, consider

$$W[f(x)] \equiv \int_0^1 dx\left(f(x)\right)^2$$

$$\delta W[\delta f(x)] \equiv \int_0^1 dx\left[\left(f(x)+\delta f(x)\right)^2 - \left(f(x)\right)^2\right]$$

$$= \int_0^1 dx\left[\cancel{\left(f(x)\right)^2} + 2f(x)\delta f(x) + \cancel{\left(\delta f(x)\right)^2} - \cancel{\left(f(x)\right)^2}\right]$$

$$= \int_0^1 dx\, 2f(x)\delta f(x) \qquad \Rightarrow \qquad \frac{\delta W}{\delta f} = 2f(x)$$

This functional derivative is valid for all $f(x)$, but it has different values for different $f(x)$. For example, consider this functional derivative evaluated for 3 different choices of $f(x)$.

$$\textit{Choice } 1: \qquad f(x)=1, \qquad K(x)=2f(x)=2 \qquad \delta W = \int_0^1 dx\, 2\delta f(x)$$

$$\textit{Choice } 2: \qquad f(x)=x, \qquad K(x)=2f(x)=2x \qquad \delta W = \int_0^1 dx\, 2x\,\delta f(x)$$

$$\textit{Choice } 3: \qquad f(x)=x^2 \qquad K(x)=2f(x)=2x^2 \qquad \delta W = \int_0^1 dx\, 2x^2\,\delta f(x)$$

The functional derivative (kernel function) $K(x)$ is different for different choices of $f(x)$.

Alternative view of functionals: A functional can be viewed as a function of N arguments, in the limit as $N \to \infty$. The differential dW = the sum of the partial derivatives, which goes over into a kernel integral:

$$\textit{Given } W(f_1, f_2, ..., f_N), \qquad dW = \frac{\partial W}{\partial f_1}df_1 + \frac{\partial W}{\partial f_2}df_2 + ... + \frac{\partial W}{\partial f_N}df_N \qquad \rightarrow$$

$$\textit{Given } W\left[f(x)\right], \quad \delta W = \int dx\,\frac{\delta W}{\delta f}\delta f(x) = \int dx\, K(x)\delta f(x) \qquad \textit{where} \quad K(x) \equiv \frac{\delta W}{\delta f}$$

Summary: A functional takes a function and produces a number. Functionals are often written in a shorthand notation which allows the functional to use differential operators on the function. A functional derivative is a *function*, which can be integrated with a small variation in the function argument to produce its change in the functional:

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$$\delta W = \int dx \frac{\delta W}{\delta f} \delta f(x) = \int dx\, K(x) \delta f(x) \qquad where \quad K(x) \equiv \frac{\delta W}{\delta f} .$$

A functional derivative treats its function argument, and derivatives of it, as independent functions, even though they're not, because of the tremendous flexibility in choosing the values of both. When a functional derivative is zero, the functional does not change with any small variations in its function argument, and also small variations in any derivatives of the function argument which the functional uses.

Reference: http://julian.tau.ac.il/~bqs/functionals/node1.html

Noether World

Noether's theorems describe a deep property of physics: the connection between symmetries and conserved quantities. Noether's theorems have had a huge impact on the framing of modern physics. They are widely misunderstood, and like so many other topics, are substantially simpler than often believed. However, Amalie Emmy Noether was not simple [www.agnesscott.edu/lriddle/women/noether.htm]. She was a preeminent mathematician, with significant accomplishments in several areas of mathematics and mathematical physics. When physicists like David Hilbert and Albert Einstein needed help with conservation laws in General Relativity, they turned to Emmy Noether. She was born in 1882 in Erlangen, Bavaria, Germany, and died in 1935 in Pennsylvania, USA.

There are three distinct theorems, with the common theme that symmetries imply constants of the motion, i.e. conserved quantities. Her first theorem describes spatial symmetries, and makes it easy to find the conserved quantity, without the burden of a specialized coordinate transformation. Her second theorem describes time symmetry, and follows trivially from the Lagrangian equations of motion. Her third theorem applies to field theories, both classical and quantum, and describes symmetries of the fields (such as gauge symmetry for electromagnetism). Not surprisingly, in quantum field theory, this third form is the most far reaching, and also the most challenging to understand.

We discuss here only the spatial and time symmetries. We show that conservation of energy (or more generally, the hamiltonian) cannot be derived in the same way as conservation of momentum.

Note that knowing the existence of constants of the motion, before solving for the motion itself, often can make solving for the motion easier. We've all seen elementary physics problems where using conservation of energy made the solution much simpler than using Newton's laws directly.

Noether's theorems require an understanding of basic Lagrangian mechanics, including ignorable coordinates and their conserved momenta.

Spatial Symmetry and Conserved Quantities

We present the ideas through an example whose properties are familiar from elementary physics, and show how Noether's theorem can derive the familiar result. We start first with spatial translation symmetry. We finish up with the much simpler time translation symmetry.

Before using one of the theorem's, you must know a symmetry of the lagrangian. Usually, they are identified by inspection of the mathematics, or by knowledge of the symmetry of the physics.

Details: Recall that if the lagrangian is independent of a particular coordinate, q_c, then the conjugate momentum is conserved (is a Constant Of the Motion):

$$\frac{\partial L}{\partial q_c} = 0 \quad \Rightarrow \quad \frac{\partial L}{\partial \dot{q}_c} \equiv p_c = COM .$$

When the lagrangian is independent of a coordinate, we say the lagrangian is symmetric w.r.t. translations in that coordinate. But our choice of coordinates is arbitrary. It is reasonable, then, to suppose that if the lagrangian has *any* spatial translation symmetry, whether it aligns with a coordinate or not, there should be a conserved quantity. We now prove this, and find the general conserved quantity, which is simply the component of the generalized momentum vector along the line of symmetry.

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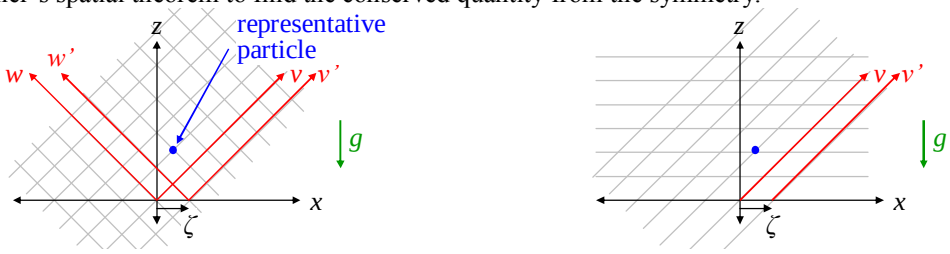
The figure below shows a 2D space with x-translation symmetry. In general, though, the coordinates used to parametrize configuration space may not align with the symmetry. We take v and w as our coordinates, though we know from elementary physics that it is x-momentum that is conserved. We now use Noether's spatial theorem to find the conserved quantity from the symmetry.



(Left) A space with x-translation symmetry. ζ parametrizes the translation.
(Right) The space labeled with (v, z) coordinates.

Noether's spatial theorem states that any *continuous* symmetry of the coordinates that leaves the lagrangian invariant corresponds to a constant of the motion, given by the theorem.

A **constant of the motion** is defined as a fixed function of the dynamic variables (coordinates and velocities) which is constant throughout the motion of the system. Common example of COMs are energy, momentum, and angular momentum. COMs are also called **conserved quantities**, and sometimes conserved "charges".

In (v, w) coordinates, our lagrangian for a particle is

$$L(v,w) = T - V = \frac{m}{2}\left(\dot{v}^2 + \dot{w}^2\right) - mg\frac{v+w}{\sqrt{2}}$$

We see by inspection that any translation of coordinates from (v, w) to (v', w') which preserves the sum ($v + w$) is a symmetry of the lagrangian. A **continuous spatial symmetry** is a coordinate transformation that can be parametrized by a single parameter, which we call ζ. The symmetry must exist for finite ζ, and often exists for unbounded ζ. In this example, our symmetry keeps ($v + w$) unchanged, so we can choose ζ to be the distance we translate the coordinate origin (passive transformation). Then

$$v' = v - \frac{\zeta}{\sqrt{2}}, \qquad w' = w + \frac{\zeta}{\sqrt{2}}, \qquad \frac{dv'}{d\zeta} = -\frac{1}{\sqrt{2}}, \qquad \frac{dw'}{d\zeta} = \frac{1}{\sqrt{2}}$$

is a continuous family of coordinate transformations, parametrized by ζ. Essentially:

We have identified a direction in configuration space
which is a symmetry of the lagrangian.

A vector **s** which points along that line of symmetry is (figure below):

$$\mathbf{s} = \frac{dq_1}{d\zeta}\hat{\mathbf{q}}_1 + \frac{dq_2}{d\zeta}\hat{\mathbf{q}}_2 + \ldots + \frac{dq_n}{d\zeta}\hat{\mathbf{q}}_n = \sum_{\sigma=1}^{n}\frac{dq_\sigma}{d\zeta}\hat{\mathbf{q}}_\sigma$$

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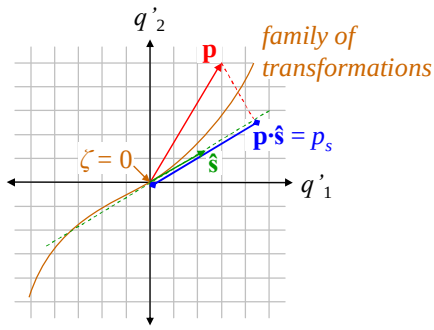

(Brown) Family of symmetry transformations. (Green) Vector in direction of symmetry. (Red) Generalized momentum vector. (Blue) Component of momentum along symmetry direction.

Note that configuration space is an abstract space, not (in general) physical space. Therefore:

The curve of symmetry in configuration space may have
a different shape than the symmetry in real space.

For example, consider 2D rotational symmetry in (r, ϕ) coordinates. In real space, the curve of symmetry is a circle, but in configuration space, the curve of increasing ϕ (i.e., the "coordinate curve") is a straight line.

It should be no surprise that the conserved quantity is the component of the generalized momentum along this line, which is simply the dot product of **s** with the generalized momentum vector:

$$\mathbf{p} = \frac{\partial L}{\partial \dot{q}_1}\hat{\mathbf{q}}_1 + \frac{\partial L}{\partial \dot{q}_2}\hat{\mathbf{q}}_2 + \ldots + \frac{\partial L}{\partial \dot{q}_n}\hat{\mathbf{q}}_n = \sum_{\sigma=1}^{n}\frac{\partial L}{\partial \dot{q}_\sigma}\hat{\mathbf{q}}_\sigma$$

$$\mathbf{p}\cdot\mathbf{s} = \frac{\partial L}{\partial \dot{q}_1}\frac{dq_1}{d\zeta} + \frac{\partial L}{\partial \dot{q}_2}\frac{dq_2}{d\zeta} + \ldots \frac{\partial L}{\partial \dot{q}_n}\frac{dq_n}{d\zeta} = \sum_{\sigma=1}^{n}\frac{\partial L}{\partial \dot{q}_\sigma}\frac{dq_\sigma}{d\zeta} = COM$$

[If you are familiar with differential geometry, you may object that the dot product above, in potentially oblique coordinates, does not use the metric. However, the momentum "vector" **p** is really a 1-form, so the dot product does not need a metric. There is an oblique coordinate example later.]

We now prove the above result. At first, we consider only infinitesimal values of ζ, and extend the result later. We constructed the transformation to leave $L(\ )$ unchanged for all ζ, so we write the total derivative $dL/d\zeta$:

$$\frac{dL}{d\zeta} = 0 = \frac{\partial L}{\partial v}\cdot\frac{dv'}{d\zeta} + \frac{\partial L}{\partial \dot{v}}\cdot\frac{d\dot{v}'}{d\zeta} + \frac{\partial L}{\partial w}\cdot\frac{dw'}{d\zeta} + \frac{\partial L}{\partial \dot{w}}\cdot\frac{d\dot{w}'}{d\zeta}$$

(1)

$$\textit{In general}: \qquad \frac{dL}{d\zeta} = 0 = \sum_{\sigma=1}^{n}\left(\frac{\partial L}{\partial q_\sigma}\frac{dq'_\sigma}{d\zeta} + \frac{\partial L}{\partial \dot{q}_\sigma}\frac{d\dot{q}'_\sigma}{d\zeta}\right)$$

If we were to change coordinates such that one coordinate $\mathbf{r}_1$ points along **s**, then the lagrangian would be independent of $\mathbf{r}_1$, and its conjugate momentum would be conserved. We now show how to compute that conserved momentum without actually bothering to find a specific coordinate transformation.

In eq. (1), as when deriving Lagrange's equations of motion, we want only one derivative w.r.t. either the q's or q-dot's. In this case, though, we expect our conserved quantity to be momentum-like, so we eliminate $\partial / \partial q$ in favor of $\partial / \partial q$-dot:

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$$\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \qquad and \qquad \frac{d\dot{q}'}{d\zeta} = \frac{d}{d\zeta}\frac{dq'}{dt} = \frac{d}{dt}\frac{dq'}{d\zeta} \qquad \Rightarrow$$

$$\frac{dL}{d\zeta} = 0 = \left(\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{v}}\right)\frac{dv'}{d\zeta} + \frac{\partial L}{\partial \dot{v}}\frac{d}{dt}\frac{dv'}{d\zeta}\right) + \left(\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{w}}\right)\frac{dw'}{d\zeta} + \frac{\partial L}{\partial \dot{w}}\frac{d}{dt}\frac{dw'}{d\zeta}\right)$$

$$In\ general: \qquad 0 = \sum_{\sigma=1}^{n}\left(\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma}\right)\frac{dq'_\sigma}{d\zeta} + \frac{\partial L}{\partial \dot{q}_\sigma}\frac{d}{dt}\frac{dq'_\sigma}{d\zeta}\right)$$

Each term in parentheses is the result of the product rule, so:

$$\frac{dL}{d\zeta} = 0 = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{v}}\frac{dv'}{d\zeta} + \frac{\partial L}{\partial \dot{w}}\frac{dw'}{d\zeta}\right] \qquad In\ general: \qquad \frac{dL}{d\zeta} = 0 = \frac{d}{dt}\sum_{\sigma=1}^{n}\frac{\partial L}{\partial \dot{q}_\sigma}\frac{dq'_\sigma}{d\zeta}$$

$$so \qquad \frac{\partial L}{\partial \dot{v}}\frac{dv'}{d\zeta} + \frac{\partial L}{\partial \dot{w}}\frac{dw'}{d\zeta} = COM = m\left(\dot{v}\left(\frac{-1}{\sqrt{2}}\right) + \dot{w}\left(\frac{1}{\sqrt{2}}\right)\right) = \frac{m}{\sqrt{2}}\left(\dot{w} - \dot{v}\right)$$

$$In\ general: \qquad \sum_{\sigma=1}^{n}\frac{\partial L}{\partial \dot{q}_\sigma}\frac{dq'_\sigma}{d\zeta} = COM$$

Let's check this result. We know that $p_x = mx$-dot is the conserved quantity. Let us convert (v, w) to the x-coordinate, and compare:

$$v = x/\sqrt{2} + other, \qquad w = -x/\sqrt{2} + other \qquad \Rightarrow$$

$$\frac{m}{\sqrt{2}}\left(\dot{w} - \dot{v}\right) = \frac{m}{\sqrt{2}}\left(\frac{-\dot{x}}{\sqrt{2}} - \frac{\dot{x}}{\sqrt{2}}\right) = -\frac{1}{2}m\dot{x}$$

Well, if (–1/2)mx-dot = COM, then mx-dot = COM. We have recovered the expected result.

So far, this works only for infinitesimal ζ. This means there exists a small neighborhood around $\zeta = 0$ where the given quantity is conserved. We now extend this result to finite ζ. There was nothing special about expanding around $\zeta = 0$. We could have expanded around any value of ζ. Now that we have a neighborhood around $\zeta = 0$ which conserves the quantity, we can choose a value of ζ away from 0, but in the neighborhood of 0, in which the quantity is still conserved. Our new neighborhood overlaps the first, but goes beyond it on one side. In this way, we can construct as many overlapping neighborhoods as we need to cover any range of ζ, out to $\pm\infty$, so long as the symmetry exists.

Note that for rotational symmetry, the overlapping neighborhoods will be along circles around the origin. Noether's theorem follows the curve of the symmetry, whatever its shape.

Example 2: We now explore some variations of this example. Suppose we defined our original transformation without the $\sqrt{2}$:

$$v' = v - \zeta, \qquad w' = w + \zeta, \qquad \frac{dv'}{d\zeta} = -1, \qquad \frac{dw'}{d\zeta} = 1$$

Our derivatives are larger by $\sqrt{2}$, so our COM is

$$m\left(\dot{w} - \dot{v}\right) = COM = -\frac{1}{\sqrt{2}}m\dot{x}$$

which differs only by a multiplicative constant, so is essentially the same result.

The conserved quantity from Noether's theorem
includes an arbitrary multiplicative factor.

Example 3: Now let's use oblique (non-orthogonal) coordinates (diagram above, right). In (v, z), we have:

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$$L(v,z) = T - V = \frac{m}{2}\left(\left(\frac{\dot{v}}{\sqrt{2}}\right)^2 + \dot{z}^2\right) - mgz$$

$$Transformation: \qquad v' = v - \zeta, \qquad z' = z, \qquad \frac{dv'}{d\zeta} = -1, \qquad \frac{dz'}{d\zeta} = 0$$

Then the conserved quantity derives only from the *v*' coordinate:

$$COM = \frac{\partial L}{\partial \dot{v}}\frac{dv'}{d\zeta} = m\frac{\dot{v}}{\sqrt{2}}(-1) \propto m\dot{x}$$

Noether's spatial theorem summary: Noether's theorems apply only to systems with a lagrangian. E.g., it does not apply to dissipative systems. That space has a symmetry tells us that *something* is conserved. Exactly *what* is conserved depends on the lagrangian. For example, for the nonrelativistic lagrangian in gravity,

$$L(x,y,z,\dot{x},\dot{y},\dot{z}) = T - V = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz \qquad \Rightarrow \qquad m\dot{x}, m\dot{y} \text{ are conserved, but not } m\dot{z}$$

For relativistic speeds, the lagrangian, and the conserved quantities, are different. L is no longer $T - V$:

$$L = -mc^2\sqrt{1 - \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)/c^2} - mgz \qquad \Rightarrow \qquad \gamma m\dot{x}, \gamma m\dot{y} \text{ are conserved, but not } \gamma m\dot{z}$$

$$where \quad \gamma \equiv \left(1 - \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)/c^2\right)^{-1/2}$$

A symmetry of the lagrangian may be more specific than a symmetry of the physics.

For example, consider physics on a table in uniform gravity, versus physics on the floor. The physics is identical: the acceleration of gravity is the same on the table and on the floor. This does *not* mean that the lagrangian is symmetric with respect to vertical translations; it is not. The lagrangian is written in terms of energy terms, and potential energy is a function of height. Therefore, vertical momentum is *not* conserved.

We often hear that spatial translation invariance implies conservation of momentum. But we could say it the other way around: the laws of physics simply *include* conservation of momentum, and therefore, the lagrangian describing those laws must include spatial translation invariance.

Time Symmetry and Conserved Quantities

What about a symmetry w.r.t. time? This is different than a spatial symmetry. Does it result in a conserved quantity? First, remembering that time and the hamiltonian (t, H) are almost like a coordinate/momentum pair, we try to apply the formula for space (derived above) to time. We see that it fails:

$$Let \qquad t' = t + \zeta, \quad \frac{dt'}{d\zeta} = 1 \qquad \Rightarrow \qquad COM = \frac{\partial L}{\partial \dot{t}}\frac{dt'}{d\zeta}, \qquad and \qquad \dot{t} = \frac{dt}{dt} = 1 \ (uh\ oh)$$

$$\Rightarrow \qquad COM = \frac{\partial L}{\partial 1}\cdot 1 \,??$$

The result is nonsense. Fortunately, the E-L equations of motion themselves furnish a simple conserved quantity, a fact which is derived in every exposition of Lagrangian mechanics:

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$$\frac{dL}{dt} = \sum_{\sigma=1}^{n}\left(\frac{\partial L}{\partial q_\sigma}\frac{dq_\sigma}{dt} + \frac{\partial L}{\partial \dot{q}_\sigma}\frac{d\dot{q}_\sigma}{dt}\right) + \frac{\partial L}{\partial t} \qquad Use: \quad \frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}$$

$$= \sum_{\sigma=1}^{n}\left(\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\frac{dq_\sigma}{dt} + \frac{\partial L}{\partial \dot{q}_\sigma}\frac{d}{dt}\dot{q}_\sigma\right) + \frac{\partial L}{\partial t}$$

$$= \frac{d}{dt}\sum_{\sigma=1}^{n}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma\right) + \frac{\partial L}{\partial t} \quad \Rightarrow \quad \frac{d}{dt}\left(\sum_{\sigma=1}^{n}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma\right) - L\right) = \frac{d}{dt}\left(\sum_{\sigma=1}^{n} p_\sigma \dot{q}_\sigma - L\right) = -\frac{\partial L}{\partial t}$$

If the lagrangian is independent of time, we have (recall $h(\cdot) \equiv$ hamiltonian written as a function of q-dot, instead of p):

$$h(q,\dot{q},t) \equiv \sum_{\sigma=1}^{n}\frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma - L = COM \qquad or \qquad H(q,p,t) \equiv \sum_{\sigma=1}^{n} p_\sigma \dot{q}_\sigma - L = COM$$

Motion With Constraints

Many types of constraints are possible on the motion of a system through configuration space, including:

1. holonomic constraints
2. differential constraints
3. integral constraints
4. velocity constraints

In this section, we discuss holonomic, differential, and velocity constraints. Holonomic constraints define relations between the coordinates, but not velocities or other dynamic quantities. All other constraints are non-holonomic. Non-holonomic constraints include differential constraints (rolling without slipping), velocity constraints, and harder things, including discontinuous constraints such as particles in a box: $0 < x < L$.

The Lagrangian formulation of mechanics allows incorporating several kinds of constraints into the equations of motion, thus solving for the constrained system. The Hamiltonian formulation does not allow constraints [Gol p335t].

There are at least 3 reasons why we solve for constrained motion:

- Even when it is possible in principle to use the constraint equations to eliminate the redundant generalized coordinates, it may be easier to keep them all, and include the constraints in the solution.
- Including all the coordinates, and using Lagrange's method of undetermined multipliers, give the *forces* of constraint. In other words, it tells how strong the constraining system has to be to enforce the constraints.
- For non-holonomic constraints, it is not generally possible to eliminate the redundant coordinates.

Holonomic Constraints

For n generalized coordinates q_i, we can write k (for "*k*onstraints") possibly time-dependent **holonomic constraints** as:

$$f_j(q_1,\ldots,q_n,t) = c_j = \text{constant}, \qquad j = 1,\ldots k \qquad \text{[F\&W 18.12 p 68, 19.4 p 69]}.$$

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["holonomic" from the Greek for "integrable" [J&S p50t].] Some authors specify the constants c_j to be zero, but we will see that only derivatives of f are used in the equations of motion, so in practice, there is no benefit to moving the constants across the equals sign.

The constraint equations imply differential relationships, obtained by taking the total time derivatives of both sides:

$$\frac{\partial f_j}{\partial q_1} dq_1 + \frac{\partial f_j}{\partial q_2} dq_2 + \ldots + \frac{\partial f_j}{\partial q_n} dq_n + \beta t = \left(\sum_{\sigma=1}^{n} \frac{\partial f_j}{\partial q_\sigma} q_\sigma \right) + \frac{\partial f_j}{\partial t} t = 0, \qquad j = 1, \ldots k .$$

There remain, then, $n - k$ *independent* degrees of motion. This means that to apply Hamilton's principle to vary the action, we can no longer vary the n q_i independently. For simplicity, consider a single constraint $f(q) = 0$, so $k = 1$. At most, we can vary $n - 1$ coordinates independently (label them q_1 through q_{n-1}), before the differential constraint fixes δq_n.

$$(1) \quad \delta q_n = -\left(\frac{\partial f}{\partial q_1} dq_1 + \frac{\partial f}{\partial q_2} dq_2 + \ldots + \frac{\partial f}{\partial q_n} dq_{n-1} \right)$$

Thus our previous derivation of Lagrange's equations of motion does not apply, because the individual coefficients in

$$\sum_{\sigma=1}^{n} \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0 \qquad \sigma = 1, \ldots n$$

need not vanish independently. I.e., the q_σ may be interdependent so as to make the sum 0 without all the coefficients being 0.

Lagrange asked, can we bring the constraints into Hamilton's principle? As noted above, we are free to take the constraint constants c_j to be zero. Then we can add some unknown, time-dependent multiple of the constraint equation, which is still zero, to the Lagrangian, without changing it:

$$L(q_i, \dot{q}_i, t) \qquad \rightarrow \qquad L(q_i, \dot{q}_i, t) + \lambda(t) f(q_i, t)$$

The $\lambda_j(t)$ are functions of time alone, just like the $q(t)$, i.e. $\partial\lambda_j/\partial q_i = 0$. [Tay p277m]

Some popular references misleadingly state that the $\lambda_j(t)$ are functions of the coordinates and/or their derivatives. This is not so, since the $\lambda(t)$ are not affected by variations $\delta q(t)$. However, when finally solving the simultaneous equations, we will obtain equations relating the $\lambda(t)$ to the $q_i(t)$ and their time derivatives, just as we obtain equations relating a given $q_i(t)$ to all the others. But as functions, the $\lambda(t)$ depend only on t.

Therefore, when taking the variation of the trajectory, the $\delta q_i(t)$ do not affect the $\lambda_j(t)$. Thus, for any variation $\delta q_i(t)$ that satisfies the constraints:

$$(2) \quad \delta S = \int_A^B dt \sum_{\sigma=1}^{n} \left(\frac{\partial L}{\partial q_\sigma} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} + \lambda(t) \frac{\partial f}{\partial q_\sigma} \right) \delta q_\sigma(t) = 0 \qquad \text{[F\&W 19.2 p69]}$$

As we freely vary the $n - 1$ independent coordinates, producing some unavoidable $\delta q_n(t)$, there must exist, then, some $\lambda(t)$, an as-yet undetermined function of time, which will cancel the forced variation of δq_n. [needs more explanation??] The $\lambda(t)$ are called **undetermined multipliers**, since we don't yet know what they are. They are also called "Lagrange multipliers."

[Many authors state that "we can now vary all of the q_i independently. I disagree. We can never vary them independently because the constraint equation always applies, and prevents it.] We can now find n equations of motion (not just $n - 1$ equations) by including the $\lambda(t)$ to cancel the forced variations of the dependent q_n.

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$$\frac{\partial L}{\partial q_\sigma} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} + \lambda(t)\frac{\partial f}{\partial q_\sigma} = 0, \qquad \sigma = 1, \dots n \qquad \text{(one constraint)}$$

We now have n equations of motion, 1 constraint equation, n unknown coordinate functions of time $q_i(t)$, and 1 unknown undetermined multiplier $\lambda(t)$. Thus we can solve the system of $n + 1$ differential equations and $n + 1$ unknowns.

For k constraints (instead of just 1), note that we can freely vary $n - k$ coordinates q_1 through q_{n-k}, and we have k forced variations of q_{n-k+1} through q_n. Then k undetermined multipliers $\lambda_j(t)$ must exist which together cancel the forced variations. This adds a summation over constraints to the equations of motion:

$$L = L(q_i, \dot{q}_i, t) + \sum_{j=1}^{k} \lambda_j(t) f_j(q_i, t) \quad \textit{where} \quad q_i \equiv \{q_1, \dots q_n\} \quad \Rightarrow$$

$$\frac{\partial L}{\partial q_\sigma} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} + \sum_{j=1}^{k} \lambda_j(t)\frac{\partial f_j}{\partial q_\sigma} = 0 \qquad \sigma = 1, \dots n \qquad (k \text{ constraints})$$

Since constraints must include forces that guide the trajectory along constraint lines [F&W p52 ff], we can include them in Lagrange's Equations of Motion (LEM). Given n coordinates and k constraints, we can rearrange the constraint equations to be additional equations of motion that give us a full system of $n + k$ equations for $n + k$ dynamic variables. With the constraints defined as above,

$$(3) \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = \sum_{j=1}^{k} \lambda_j(t)\frac{\partial f_j}{\partial q_\sigma} \equiv Q_\sigma(t)_{,\,constraint} \qquad \sigma = 1, \dots, n \qquad \text{[F\&W 19.3 p69]}$$

which can be thought of as "rate of change of momentum − forces due to potentials = other forces." The other forces are exactly the n generalized forces of constraint $Q_\sigma(t)_{,constraint}$, as functions of time.

These n equations plus the k constraint equations solve for the n $q_\sigma(t)$ and the k $\lambda_j(t)$.

In the course of solving the equations, the $q_i(t)$ and $\lambda_j(t)$ will have relations to the coordinates, including their time derivatives possibly up to order $2(n - k)$, and possibly t. You solve for the $q_i(t)$ and $\lambda_j(t)$ simultaneously from the equations of motion, and the constraint equations. The solution can lead to relations including such higher order derivatives of $q_i(t)$.

Generalized forces satisfy the same momentum and work equations as Cartesian forces:

$$\delta W = Q_i\ \delta q_i \qquad \text{(no sum on } i\text{)}, \qquad i = 1, \dots n \qquad \text{[from F\&W 15.4 p54]}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \equiv \dot{p}_i = \frac{\partial L}{\partial q_i} + Q_{i,\,constraint}(t) \equiv Q_i(t), \qquad i = 1, \dots, n \qquad \text{[from (3) above]}$$

Differential Constraints

Differential constraints are written relating the differentials of coordinates, the coordinates, and possibly time:

$$\alpha_{j1}(q,t)\, dq_1 + \alpha_{j2}(q,t)\, dq_2 + \dots + \alpha_{jn}(q,t)\, dq_n + \beta_j\ dt = \sum_{\sigma=1}^{n} \alpha_{j\sigma}(q,t)\, dq_\sigma + \beta(q,t)\, dt = 0$$

$$j = 1, \dots k \qquad \textit{where} \quad q \equiv \{q_1, \dots q_n\} \qquad \text{[F\&W p71m]}$$

The $\alpha_{j\sigma}(q_i, t)$ are functions of the coordinates and possibly time.

Recall that for holonomic constraints, in applying Hamilton's principle (eq. (2)), we used only the resulting differential constraint, Eq (1), where the $\partial f/\partial q_i$ are also functions of the coordinates and time. Therefore, differential constraints are also included in that part of the holonomic constraint analysis. With a simple change of notation to the coefficients $\alpha_{j\sigma}$, we have:

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$$\frac{\partial f_j(q_i,t)}{\partial q_\sigma} \to \alpha_{j\sigma}(q_i,t): \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = \sum_{j=1}^{k} \lambda_j(t)\alpha_{j\sigma}(q_i,t) \equiv Q_\sigma(t) \qquad \sigma = 1, \dots n$$

[F&W p71b]

However, we still need k constraint equations to augment the n equations of motion, to yield $n + k$ equations in all, needed to solve for n unknown $q_i(t)$ and k unknown $\lambda_j(t)$. We get these by dividing the differential constraint equation by dt. Physically, this means the instantaneous velocities of the coordinates must obey the relationship:

$$\alpha_{j1}(q_i,t)\dot{q}_1 + \alpha_{j2}(q_i,t)\dot{q}_2 + \dots + \alpha_{jn}(q_i,t)\dot{q}_n + \beta_j = \sum_{\sigma=1}^{n} \alpha_{j\sigma}(q_i,t)\dot{q}_\sigma + \beta_j(q_i,t) = 0$$

$$j = 1, \dots k \qquad \text{[F\&W p71b]}$$

[F&W p71b] note that this can be extended to the case where the α_j are functions of the velocities q-dot$_i$, but give no references. Velocity constraints (described below) are tricky.

Constraints Including Velocities

The topic of velocity constraints has been fraught with much error and confusion, even among physicists. We explain here several of the errors, and their corrections, citing published works along the way.

Constraints including velocities are one kind of non-holonomic constraints. We can extend the holonomic constraints by including the velocities in the constraint functions:

$$g_j(q_1, \dots q_n, \dot{q}_1, \dots \dot{q}_n, t) = c_j = \text{constant} \qquad j = 1, \dots k$$

Note that [Gol] has a serious error concerning velocity constraints.

[Thanks to Patrick Geisler for pointing this out to me.] The error is based on a 1966 paper [Ray1], which was retracted the same year by the author [Ray2]. ([Gol p47b] also mistakenly cites the year as 1996.) Specifically, [Gol p47] (following [Ray1]) uses the method of Lagrangian multipliers with an incorrect trajectory variation procedure, yielding an incorrect equation of motion. A corrected, but still incomplete, procedure is given by [S&C1], [S&C2], and [J&S].

The error in [Ray1] and [Gol] derives from allowing variational trajectories which satisfy the constraints, but which cannot be achieved by any physical constraining system. Specifically, for the constraining mechanism to achieve the forces necessary for the trajectory, it would have to exert forces that are not perpendicular to the motion, i.e. would have to violate D'Alembert's Principle. In other words, the constraining mechanism would have to "drag" the system in unphysical ways. Furthermore, even with such a magical capability, the solution is not unique, because the equations are under-determined [J&S??]. Clearly, any given physical system, with a full set of initial conditions, has only one (unique) motion resulting from it.

[J&S p116t] claim that the correct variational procedure allows only trajectories which satisfy d'Alembert's principle (which is disputed by [Liu]). Surprisingly, such a variational procedure is very simple. Instead of derivatives of the constraint equations with respect to the coordinates, we take derivatives with respect to the velocities. We quote the result, which apparently even [J&S] are uneasy about, since they only timidly endorse it, saying "this result is generally accepted...." [J&S p116t]:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = \sum_{j=1}^{k} \lambda_j(t)\frac{\partial g_j}{\partial \dot{q}_\sigma} \equiv Q_\sigma(t)_{,\,constraint} \qquad \sigma = 1, \dots, n \qquad \text{[from J\&S 3.21 p116t]}.$$

It may seem that in taking the derivative of g_j, we have lost information, but recall the that full constraint equation is part of the set of equations used to finally solve the motion. Hence the information is retained.

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However, the above equation is incomplete, because some of the k constraints may include *only* coordinates (but not velocities), while others may include both [Liu (12) p752, and following]. How do we simultaneously include both kinds of constraints? Furthermore, this result is based on d'Alembert's principle , which [Liu] claims is not valid for *arbitrary* velocity constraints, but only for a restricted class of them.

Surprisingly, [Liu] addresses both of these problems in a 1981 paper that far predates [J&S]. Liu's approach uses $F = ma$ directly, and therefore converts the original constraints into constraints on *acceleration*. Liu's prescription is:

For each constraints of the form: $f_j(q,t) = c_j$ *define* $h_j(q,\dot{q},\ddot{q},t) \equiv \dfrac{\partial^2 f_j(q,t)}{\partial t^2}$

For each constraints of the form: $g_j(q,\dot{q},t) = c_j$ *define* $h_j(q,\dot{q},\ddot{q},t) \equiv \dfrac{\partial g_j(q,\dot{q},t)}{\partial t}$

By construction, the $h_j(\cdot)$ (j = 1, ... k) form a set of k constraints, all of which are *linear* in the accelerations. As before, the information lost in taking derivatives here is retained in the constraint equations which form part of the complete set of simultaneous equations that must be solved for the motion. Since [Liu] does not use Hamilton's (or any other) variational principle, his most general equations of motion are Newtonian and in 3D vector form. For N particles (and therefore $3N - k$ independent degrees of freedom):

$$m_i\ddot{\mathbf{r}}_i = \mathbf{F}_i + \sum_{j=1}^{k} \lambda_j(t)\frac{\partial h_j}{\partial \ddot{\mathbf{r}}_i}, \quad i = 1, \ldots N \qquad \text{[Liu (15) p752]}.$$

For those classes of constraints (described later) which satisfy d'Alembert's principle, and therefore also Hamilton's principle, we can use generalized coordinates

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Q_{\sigma,\text{constraint}}(t) = \sum_{j=1}^{k} \lambda_j(t)\frac{\partial h_j}{\partial \ddot{q}_\sigma} \qquad \text{[From Liu (19) p752]}.$$

[Liu sec. 6 p753] derives d'Alembert's principle, concluding:

d'Alembert's principle is valid if and only if the constraints on the system are either (i) holonomic, or (ii) homogeneous in velocity dependence.

By "homogeneous in velocity dependence" he means the velocity terms, taken separately from the holonomic terms, are homogeneous (of any order) in the velocities. In other words, the constraint can be written as:

$$g(q_i,\dot{q}_i,t) = f(q_i,t) + s(q_i,\dot{q}_i,t) \qquad \textit{where} \quad s(q_i,\dot{q}_i,t) \text{ is homogeneous in the } q_i$$

Hamilton's principle is valid only when d'Alembert's principle is valid.

[Liu p752] points out that Newton's 2nd law, $F = ma$, applies to all types of constraints, even when Hamilton's principle does not. Furthermore:

Newtonian-like equations of motion can be solved for the forces of constraint, even when Hamilton's principle does not apply.

Summary

We have shown that holonomic constraints can use the method of Lagrange undetermined multipliers to solve simultaneously for all n of the interdependent $q_i(t)$, and for the undetermined multipliers $\lambda_j(t)$. This also provides the forces of constraint. Similarly, for differential constraints, the method of undetermined multipliers solves for the motion and the forces of constraint, but uses a velocity relationship to supplement Lagrange's equations of motion.

Another variant of undetermined multipliers can be used to solve for integral constraints [Aro ch 8].

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Velocity constraints have been widely misunderstood and incorrectly published by physicists for a century. The most definitive work seems to be [Liu] in 1981, though his paper is only 4 pages. The final result converts each constraint equation, whether a holonomic or velocity constraint, into an acceleration constraint, which is necessarily linear in the accelerations. Undetermined multipliers once again provide the final mathematical step to solve for the motion and constraint forces. d'Alembert's principle, and therefore Hamilton's principle of stationary action, is only valid for holonomic constraints, and velocity constraints whose velocity terms are homogeneous (of any order) in the velocities.

Newtonian-like equations of motion can be solved for the forces of constraint, even when Hamilton's principle does not apply.

Hamiltonian Mechanics

Hamiltonian mechanics rewrites lagrangian mechanics in a different form, which is useful for many applications. In particular, quantum mechanics starts with the Hamiltonian form of classical mechanics.

The hamiltonian is *defined* as

$$H(q_i, p_i, t) \equiv \left(\sum_{i=1}^{\#\text{ coordinates}} p_i \dot{q}_i \right) - L(q_i, \dot{q}_i, t), \quad \textit{where} \quad p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \qquad \text{[F\&W 20.12 p79b]}.$$

where L is any lagrangian appropriate for the physics. Note that the hamiltonian is a function of coordinates and momenta, whereas the Lagrangian is a function of coordinates and their velocities. Therefore, when transforming from the lagrangian to the hamiltonian, one must eliminate the velocities in favor of the momenta using the definition of **conjugate momentum**, included above.

The units of the hamiltonian are always energy.
The units of a coordinate times its conjugate momentum are always action (energy-time).

This follows from the lagrangian having units of energy, and the definition of momentum above.

In the case of time invariant potentials, time invariant generalized coordinate transformations, and nonrelativistic $L = T - V$, the above definition reduces to $H = T + V$. [Since some reference address only this latter case in earlier discussions, they confusingly chose to define their hamiltonian in a highly nonstandard way [P&R 4.6 p43, though they fix it later in the book.]]

The hamiltonian is *not* defined as $T + V$,
because the above definition allows for constrained velocities and time-dependent potentials.

[TBS: Bead on rotating hoop example. Linear bead on an accelerating (oscillating?) spring example.] Note that constrained velocities usually require that the constraints do work on the system, thus changing its energy over time.

Note that the value of the lagrangian function is always independent of the chosen generalized coordinates, since it is defined by energy terms which are coordinate independent. In contrast [Gol p63t],

The values of the hamiltonian function *depend* on the coordinates chosen.

This can be seen from the defining equation above, which explicitly includes the coordinates and momenta, along with the coordinate independent lagrangian.

It is not clear that the hamiltonian is well-defined for a dissipative system. Hamiltonian mechanics is generally used only for non-dissipative systems (which includes microscopic systems).

The energy function: We can write the hamiltonian as a function of the q_i and their velocities [Gol ??]:

$$h(q_i, \dot{q}_i, t) \equiv \sum_{i=1}^{n} p_i \dot{q}_i - L(q_i, \dot{q}_i, t) = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L(q_i, \dot{q}_i, t).$$

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However, this is only the energy of the system in some cases, which we detail later.

TBS: How to hide time dependence in a system. For example, to add time-dependence to a given mechanical system, we could define a new generalized coordinate, call it q_t, with a constrained velocity and initial condition:

$$\dot{q}_t = 1,\ q_t(0) = 0 \quad \Rightarrow \qquad q_t(t) = t$$

and then claim the hamiltonian is time-independent. The changes in energy that result are now due to the force which is enforcing the constrained velocity on q_t.

When Does the Hamiltonian Equal Energy, and Other Questions?

There are three distinct questions we might ask about the hamiltonian:

1. Is the hamiltonian "conserved," i.e. a constant of the motion?
2. Does the hamiltonian equal the total energy?
3. Does the hamiltonian equal $T + V$?

Note that [Gol p345]

H may be the total energy, or not. H may be conserved, or not. These are independent properties. A given H may be either, both, or neither.

It is not clear (to me, at least) that the hamiltonian is well-defined for a dissipative system.

1. Is the hamiltonian conserved? This is simple: if the lagrangian does not explicitly depend on time, and all forces derive from a potential, then the hamiltonian is conserved:

$$\frac{\partial L(q,\dot{q},t)}{\partial t} = 0 \qquad \Rightarrow \qquad L = L(q,\dot{q}) \qquad \textit{and} \qquad \frac{dH(q,p)}{dt} = 0, \text{ i.e. } H \text{ is conserved}.$$

This follows directly from the equations of motion [F&W p80t]. Note that the modified equations of motion which include friction (or drag) do *not* conserve the hamiltonian. Note that explicit time dependence in the lagrangian is equivalent to explicit time dependence in the hamiltonian. Also, time-dependent potentials lead to time-dependent lagrangians (and hamiltonians), and non-conservation of H.

2. Does the hamiltonian equal the energy? Sometimes the hamiltonian is the total energy; sometimes it's not. How to tell the difference? There are at least 4 equivalent ways to insure that H = total energy, but as we show next, they are not *necessary*:

- When H = total energy, i.e., when you work out total energy as a function of the dynamic variables (q, q-dot) or (q, p), and see that it equals $H(q, p)$. This may seem trivial, but it works.
- When the generalized coordinates include all *motion* of the system, and all the forces derive from a potential that is velocity independent [Gol p345]. This will lead to the preceding case.
- When T (kinetic energy) is quadratic in the generalized velocities, i.e. is a homogeneous function of degree 2 of the velocities [Arovas], and all the forces derive from a potential that is velocity independent [Gol p345]. This homogeneity implies that *all* the motion of the system is included by the generalized coordinates, which is the preceding case above. Note that the coefficients of the quadratic velocity terms may be functions of the coordinates (i.e., need not be constant).
- When transformation of generalized coordinates to Cartesian coordinates does not explicitly include the time, and all the forces derive from a potential that is velocity independent [Gol p345].

Note that total energy may be different than $T + V$ if $V = V(q, q\text{-dot})$ includes velocities (see below).

Only conservative systems (no friction or drag) allow the hamiltonian to be the total energy.

Time independent constraints are insufficient to make the hamiltonian be the total energy.

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In some cases, even though the lagrangian includes terms linear in velocities, the hamiltonian is *still* the total energy. This happens with magnetic forces (more later).

3. Does *H* = *T* + *V*? This depends on the author's conventions. Some authors include velocity dependent potentials, such as the magnetic vector potential, in *V*, which then becomes a function of positions and velocities:

$$V = V(q, \dot{q}) \qquad \text{velocity dependent forces}$$

For example, for a charged particle subject to magnetic forces we have (SI units)

$$L(q,\dot{q}) = T - U(q) + e\dot{\mathbf{q}} \cdot \mathbf{A}(q) \qquad \textit{where} \quad \dot{\mathbf{q}} \equiv (\dot{q}_1, \ldots \dot{q}_n),$$
$$\mathbf{A} \equiv \text{magnetic vector potential}, \quad e \equiv \text{electric charge}$$
$$U(q) \equiv \text{position dependent potential}$$

However, some authors include the magnetic term in the potential, yielding a position and velocity dependent potential *V*(*q*, *q*-dot) [Aro 7.69]:

$$V(q,\dot{q}) \equiv U(q) - e\dot{\mathbf{q}} \cdot \mathbf{A} \qquad \Rightarrow$$
$$L(q,\dot{q}) = T(\dot{q}) - V(q,\dot{q}) \qquad \text{(combined position-velocity potential)}$$

In this case, *V* is a "potential," but it is no longer a "potential energy," since it includes a term which is not an energy. Then *H* does not equal *T* + *V*(*q*, *q*-dot), but (as shown below) it *does* equal the total energy.

All Charged Up: The Magnetic Hamiltonian

It is interesting to compare the hamiltonian of a charged particle in a magnetic field to the total energy. Many references say the hamiltonian is only the total energy if the "potential" is independent of velocity. This statement is ambiguous, since it is not clear if the magnetic term in the lagrangian is considered a "potential." Regardless, the magnetic lagrangian term is linear in the velocity.

We compute the hamiltonian of a particle of charge *e* in a magnetic field, in 3D space, starting with the lagrangian (in SI units):

$$L(\mathbf{x},\dot{\mathbf{x}}) = \frac{m}{2}\dot{\mathbf{x}}^2 + e\dot{\mathbf{x}} \cdot \mathbf{A} - V(\mathbf{x}), \qquad \mathbf{p} \equiv \frac{\partial L}{\partial \dot{\mathbf{x}}} = m\dot{\mathbf{x}} + e\mathbf{A} \qquad \Rightarrow \qquad \dot{\mathbf{x}} = \frac{\mathbf{p} - e\mathbf{A}}{m}$$

$$H(\mathbf{x},\mathbf{p}) \equiv \mathbf{p} \cdot \dot{\mathbf{x}} - L = \mathbf{p} \cdot \frac{\mathbf{p} - e\mathbf{A}}{m} - \frac{m}{2}\left(\frac{\mathbf{p} - e\mathbf{A}}{m}\right)^2 - e\left(\frac{\mathbf{p} - e\mathbf{A}}{m}\right) \cdot \mathbf{A} + V(\mathbf{x})$$
$$= \left(\frac{\mathbf{p} - e\mathbf{A}}{m}\right) \cdot \left(\mathbf{p} - \frac{m}{2}\left(\frac{\mathbf{p} - e\mathbf{A}}{m}\right) - e\mathbf{A}\right) + V(\mathbf{x})$$
$$= \left(\frac{\mathbf{p} - e\mathbf{A}}{m}\right) \cdot \left(\frac{\mathbf{p} - e\mathbf{A}}{2}\right) + V(\mathbf{x}) = \frac{(\mathbf{p} - e\mathbf{A})^2}{2m} + V(\mathbf{x}) = \text{total energy}$$

This reflect the fact that "kinetic momentum" $\mathbf{p}_{kin} = m\mathbf{v} = \mathbf{p} - e\mathbf{A}$, and kinetic energy is $(\mathbf{p}_{kin})^2/2m$. However:

Even though the magnetic lagrangian includes a velocity dependent term outside the kinetic energy, the hamiltonian *is* indeed the total energy.

Canonical Coordinates

Briefly, a **canonical coordinate** is a *continuous* real value that describes the position or orientation of some part of the mechanical system. The **conjugate momentum** is the partial derivative of the lagrangian with respect to the coordinate velocity:

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$$p \equiv \frac{\partial L(q,\dot{q},t),}{\partial \dot{q}} \quad or \quad p_\sigma \equiv \frac{\partial L(q_i,\dot{q}_i,t),}{\partial \dot{q}_\sigma} \qquad \sigma = 1,\ldots \#\text{ degrees of motion}.$$

A **canonical pair** is the matched pair of a canonical coordinate with its conjugate momentum (q_σ, p_σ).

Example of a *non*-canonical coordinate: angle mod 2π. It fully defines the physical configuration (position) of the system, but cannot be used to analyze its dynamics, because it jumps discontinuously from 2π to 0 when wrapping around to the starting point. To be a valid coordinate, it must continue on with values > 2π, so that its derivative is physically meaningful.

Are Coordinates Independent?

The following considerations are important to understanding Poisson brackets, as well as general Lagrangian and Hamiltonian mechanics. They extend the above section "What Is the Derivative With Respect To a Derivative?"

In Hamiltonian mechanics, we say things like $\frac{\partial p_a}{\partial q_b} = 0$, because p_a and q_b are "independent." What does this mean? Given a system, we can (in principle) solve it to find $p_a(t)$ and $q_b(t)$. Then in some infinitesimal time interval dt, p_a will change by an amount $dp_a = \dot{p}_a(t)\,dt$, and q_b will change by an amount $dq_b = \dot{q}_b(t)\,dt$. So you might be mislead into thinking that we could define the derivative dp_a / dq_b as the ratio

$$\frac{dp_a}{dq_b} = \frac{\dot{p}_a(t)\,dt}{\dot{q}_b(t)\,dt} = \frac{\dot{p}_a(t)}{\dot{q}_b(t)} \neq 0 .$$

But in this case, the "derivative" $\frac{dp_a}{dq_b}$ is ill-defined. $p_a(t)$ is not a function of q_b, so how can it have any derivative with respect to q_b? Clearly, when we say that p_a and q_b are "independent," we do *not* mean that in the actual motion, one can change without the other changing.

So what does $\frac{\partial p_a}{\partial q_b} = 0$ mean? Let us approach this through some simple examples. Consider a function

$$X(c,d) = c^2 + d^3. \qquad Then \quad \frac{\partial X}{\partial c} = 2c \qquad and \quad \frac{\partial X}{\partial d} = 3d^2$$

We can avoid introducing the spurious function X by saying more directly,

$$\frac{\partial}{\partial c}\left(c^2 + d^3\right) = 2c \qquad and \qquad \frac{\partial}{\partial d}\left(c^2 + d^3\right) = 3d^2$$

On the other hand, X is a *function*, and we could describe this same function with different variables as its arguments:

$$X(e,f) = e^2 + f^3. \qquad Then \quad \frac{\partial X}{\partial e} = 2e \qquad and \quad \frac{\partial X}{\partial f} = 3f^2$$

This new description has not changed the *function* X; we've simply changed the names of its arguments. And we could still eliminate X entirely by writing

$$\frac{\partial}{\partial e}\left(e^2 + f^3\right) = 2e \qquad and \qquad \frac{\partial}{\partial f}\left(e^2 + f^3\right) = 3f^2$$

Let us change argument names again, and define

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$$X(q_b, p_a) = q_b{}^2 + p_a{}^3. \qquad \textit{Then} \quad \frac{\partial X}{\partial q_b} = 2q_b \qquad \textit{and} \quad \frac{\partial X}{\partial p_a} = 3p_a{}^2.$$

$$\textit{Or} \quad \frac{\partial}{\partial q_b}\left(q_b{}^2 + p_a{}^3\right) = 2q_b \qquad \textit{and} \quad \frac{\partial}{\partial p_a}\left(q_b{}^2 + p_a{}^3\right) = 3p_a{}^2$$

Now consider a simpler function X:

$$X(q_b, p_a) = p_a. \qquad \textit{Then} \quad \frac{\partial X}{\partial q_b} = 0 \qquad \textit{and} \quad \frac{\partial X}{\partial p_a} = 1.$$

$$\textit{Or} \quad \frac{\partial}{\partial q_b}\left(p_a\right) = \frac{\partial p_a}{\partial q_b} = 0 \qquad \textit{and} \quad \frac{\partial}{\partial p_a} p_a = \frac{\partial p_a}{\partial p_a} = 1$$

The first equation in the last line is the desired result. It means that, when "p_a" is viewed *as a function of the dynamic variables*, rather than as $p_a(t)$ [a function of time], then $\frac{\partial p_a}{\partial q_b} = 0$.

Similarly, the 2nd equation of the last line says that, when "p_a" is viewed *as a function of the dynamic variables*, rather than as $p_a(t)$ [a function of time], then $\frac{\partial p_a}{\partial p_a} = 1$.

Lagrangian and Hamiltonian Relations

$$L \equiv T - V, \qquad p_\sigma \equiv \frac{\partial L}{\partial \dot{q}_\sigma}, \qquad H \equiv \sum_{\sigma=1}^{\#\,coordinates} p_\sigma \dot{q}_\sigma - L, \qquad \dot{p}_\sigma = -\frac{\partial H}{\partial q}, \qquad \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma}$$

[F&W 32.9, 32.11 p 175; 32.15a,b p 176; 32.29, 32.30 p 178]

The hamiltonian depends on time only if it is *explicitly* time dependent:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \qquad \text{[F\&W 32.18 p 177]}$$

A conservative system implies a potential $V(\mathbf{q})$ a function of coordinates only (but not time), and holonomic time-*independent* constraints:

$$H(\mathbf{q}, \mathbf{p}) = T(\mathbf{q}, \mathbf{p}) + V(\mathbf{q}) = Energy = const$$

(conservative $\Rightarrow$ holonomic, time-independent potential & constraints) [F&W 32.20 p 177]

Transformations

Point transformations involve only the coordinates and time (not velocities or momenta):

$$q_\sigma = q_\sigma\left(Q_1, ..., Q_n, t\right) \qquad \text{[F\&W 32.4a p 174]}$$

Canonical transformations involve coordinates and momenta for every transformed variable:

$$p_\sigma = p_\sigma\left(P_1, ..., P_n, Q_1, ..., Q_n, t\right) \qquad \text{[F\&W 34.1a,b p181]}$$

$$q_\sigma = q_\sigma\left(P_1, ..., P_n, Q_1, ..., Q_n, t\right)$$

The central force transformation from 2-body problem to the 1-body problem is an example of a canonical transformation. In this case, the new canonical equations of motion separate into independent equations for the center of mass motion and the orbiting motion. That's why it's so useful. Usually we can ignore the center-of-mass motion, and focus only on the orbiting motion.

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Canonical transformations preserve Hamilton's Eqs of Motion (HEM), phase-space volume, and Poisson brackets → transformation is canonical if and only if it has unit Jacobian (P&R 6.54 p 94):

$$\frac{\partial(Q,P)}{\partial(q,p)} = \begin{Vmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{Vmatrix} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = 1$$

Generating functions are functions of one old variable and one new variable. Time-independent generating functions are $F_1(Q, q)$, $F_2(P, q)$, $F_3(Q, p)$, $F_4(P, p)$. [P&R Table 6.1 p 90]:

$$q = -\frac{\partial F}{\partial p}, \qquad p = \frac{\partial F}{\partial q}, Q = \frac{\partial F}{\partial P}, P = -\frac{\partial F}{\partial Q}, \qquad K(Q,P) = \tilde{H}(Q,P) = H(q,p)$$

Time-dependent generating functions: $F_1(Q, q, t)$, $F_2(P, q, t)$, $F_3(Q, p, t)$, $F_4(P, p, t)$. [P&R , 6.60 p 95, 6.70 p 97] F&W's $S(q, P, t)$ is like P&R's $F_2(P, q)$ (F&W35.4a, b p 184):

$$\frac{\partial F_1}{\partial t} = \frac{\partial F_2}{\partial t} = \frac{\partial F_3}{\partial t} = \frac{\partial F_4}{\partial t}, \qquad K(Q,P,t) = \tilde{H}(Q,P,t) = H(q,p,t) + \frac{\partial F_j}{\partial t}$$
$$S(q,P,t) = F_2(P,q,t)$$

How does the new lagrangian relate to the old lagrangian??

$$H(q,p) \to K(Q,P) \qquad \Rightarrow \qquad L(q,\dot{q},t) = \sum p\dot{q} - H = L(Q,\dot{Q},t) = \sum PQ - K(Q,P) + \frac{dF}{dt}$$

where $F = F_1(q,Q,t)$

$$F = F_2(q,P,t) - \sum Qp$$

$$F = F_3(p,Q,t) - \sum qp$$

$$F = F_4(p,P,t) + \sum qp - \sum QP$$

For example, for $F_1(q, Q, t)$:

$$\sum p\dot{q} - H = \sum P\dot{Q} - K + \frac{dF_1}{dt} = P\dot{Q} - K + \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial Q}\dot{Q}$$

$$\Rightarrow \qquad p = -\frac{\partial F}{\partial q}, \qquad P = -\frac{\partial F}{\partial Q}, \qquad K = H + \frac{\partial F}{\partial t}$$

Interesting and Useful Canonical Transforms

[From PD 26c]

a) Identity: $F_2(q,P) = qP \quad \Rightarrow \quad p = P, \quad Q = q$

b) Interchange: $F_1(q,Q) = qQ \quad \Rightarrow \quad p = Q, \quad P = -q$

Graphically demonstrates the interchangeability of 'p' and 'q' labels in Hamiltonian mechanics.

c) Point transform: $F_2(q,P) = f(q)P \Rightarrow \quad p = P\frac{df}{dq}, \qquad Q = f(q)$

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d) Orthogonal transform: $F_2(q,P) = \sum_{i,k} P_i a_{ik} q_k \quad \Rightarrow \quad p_i = \sum_k a_{ki} P_k, \quad Q = \sum_k q_k a_{ik}$

Hamilton-Jacobi Theory

The goal is to transform so all the Q's and P's are constant, achieved by transforming H to zero. F&W's $S(q, P, t)$ is like P&R's $F_2(P, q)$ [F&W35.4a, b p 184, 35.12 p 185]:

$$p = \frac{\partial S}{\partial q},\ Q = \frac{\partial S}{\partial P},\ K = \tilde{H}(P,Q,t) = H(p,q,t) + \frac{\partial S}{\partial t} = 0$$

$$\Rightarrow \qquad H\left(\frac{\partial S}{\partial q_1},\ldots,\frac{\partial S}{\partial q_n}, q_1,\ldots,q_n,t\right) + \frac{\partial S}{\partial t} = 0$$

$$\Rightarrow \qquad P_\sigma = const = \alpha_\sigma \quad and \quad Q_\sigma = const = \beta_\sigma = \frac{\partial S}{\partial P_\sigma} = \frac{\partial S}{\partial \alpha_\sigma}$$

where all the P's and Q's are constant, and we rewrite them as α and β [F&W 35.15, 19, 20 p 186].

$S(t)$ evaluated *along the actual trajectory* of the system is the action [F&W 35.23 p 187]:

$$\frac{dS}{dt} = L$$

Time-Independent H-J Equation

If H has no explicit time dependence, then $H = const = E$, then we define W, and compute H (F&W 35.25, 26 p 187, but I think the α arguments for W should start with α_2):

$$S(q_1,\ldots,q_n,\alpha_1,\ldots,\alpha_n,t) = W(q_1,\ldots,q_n,\alpha_2,\ldots,\alpha_n) - \alpha_1 t$$

$$H(p,q,t) + \frac{\partial S}{\partial t} = 0 \qquad \Rightarrow \qquad H\left(\frac{\partial W}{\partial q_1},\ldots,\frac{\partial W}{\partial q_n}, q_1,\ldots,q_n,t\right) = -\frac{\partial S}{\partial t} = \alpha_1 = E$$

Then if W separates in the q's, we have some hope of actually solving for W, then finding S from W, and finally computing the solution to the original problem from S, noting that the constants α_σ and β_σ are determined from initial conditions (F&W 35.27 p 188, 36.1 p 191, cf. 35.38-47 p 189-190, 35.14 p 186):

$$W(q_1,\ldots,q_n,\alpha_1,\ldots,\alpha_n) = W_1(q_1,\alpha_1,\ldots,\alpha_n) + W_2(q_2,\alpha_1,\ldots,\alpha_n) + \ldots + W_n(q_n,\alpha_1,\ldots,\alpha_n)$$

$$S(q_1,\ldots,q_n,\alpha_1,\ldots,\alpha_n,t) = W(q_1,\ldots,q_n,\alpha_1,\ldots,\alpha_n) - Et$$

$$\beta_\sigma = Q_\sigma = const = \frac{\partial S(q\ldots,\alpha\ldots,t)}{\partial \alpha_\sigma} = \frac{\partial S(q\ldots,P\ldots,t)}{\partial P_\sigma}$$

$$Inverting \qquad \Rightarrow \qquad q_\sigma(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,t)$$

$$and \qquad p_\sigma = \frac{\partial S(q_1,\ldots,q_n,\alpha_1,\ldots,\alpha_n,t)}{\partial q_\sigma} \qquad or \qquad p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

Poisson brackets [F&W 37.1 p 197]:

$$\{F,G\} \equiv \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma}\frac{\partial G}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma}\frac{\partial G}{\partial q_\sigma}\right), \qquad \frac{dA(\mathbf{q},\dot{\mathbf{q}},t)}{dt} = \{A,H\} + \frac{\partial A}{\partial t}$$

Action-Angle Variables

For the bounded motion of conservative systems, the motion is periodic. The simplest general variables are action-angle variables.

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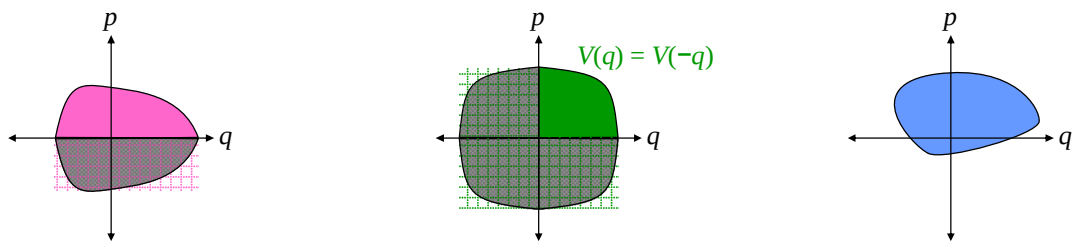

Periodic Motion in an asymmetric Potential | Periodic Motion in a Symmetric Potential | Arbitrary Periodic Motion

Phase curves for three kinds of periodic motion: (Left) Asymmetric potential is still symmetric about 'q' axis. (Center) Symmetric potential is symmetric about both axes. (Right) Non-symmetric motion is only possible with velocity dependent forces.

Thus, the goal is to transform to canonical momenta $I = const$ (action), and coordinates θ (angle variables) that linearly increase with time. This implies the hamiltonian $H(I)$ is independent of θ [P&R 7.3-6 p 103-105, PD p 30]:

$$\dot{I} = -\frac{\partial H(I)}{\partial \theta} = 0 \quad \Rightarrow \quad I = COM \text{ (constant of motion)},$$

$$\dot{\theta} = \omega(I) = \frac{\partial H(I)}{\partial I} = \frac{\partial E(I)}{\partial I} = COM \quad \Rightarrow \quad \theta(t) = \omega(I)t + \delta$$

F&W say the "angles" are *not* canonical conjugates of the action [F&W 36.13 p 193], but PD says they're wrong. The above eqs are Hamilton's, and [P&R p 106 and many others] and [PD p 33] say "the transformation $(q, p) \rightarrow (\theta, I)$ is canonical".

The action variable has dimensions of action = angular momentum = energy-time. The angle variable is dimensionless (P&R p 107). For libration (oscillation), action can be computed from E & V [P&R 7.10-14 p 106-107]:

$$H(q,p) = E = \frac{p^2}{2m} + V(q) \quad \Rightarrow \qquad p(q,E) = \sqrt{2m\left(E - V(q)\right)}$$

$$I(E) = \frac{\text{phase-space-area}}{2\pi} = \frac{1}{2\pi}\iint dq\, dp = \frac{1}{2\pi}\oint dq\; p(q,E) \quad \text{[arbitrary phase curve]}$$

$$= \frac{1}{\pi}\int_{q-\min}^{q-\max} dq\, \sqrt{2m\left(E - V(q)\right)} \qquad \text{[particle in simple potential]}$$

$$= \frac{2}{\pi}\int_{0}^{q-\max} dq\, \sqrt{2m\left(E - V_e(q)\right)} \qquad \text{[particle in symmetric potential } V_e(q) = V_e(-q)\text{]}$$

The turning points q_{min} and q_{max} are where $T = 0$, or $E = V$, or *velocity* = 0.

Abbreviated action is the generating function for action-angle canonical transform [P&R 7.29 p 112]:

$$F_2(q,I) = S_2(q,I) = \int_0^q dq\; p(q,I)$$

For periodic motion (rotations), simply change the above integration limits to one period, and rename $q = \psi$ (P&R 7.44, 7.46 p 114, cf. 7.51 p 115). However, P&R *assume* a period of 2π in the *original* coordinate ψ (not just the transformed coordinate θ), which is bad (e.g., P&R problem 7.2: original period is 4π):

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$$H(q=\psi,p)=H(\psi+2\pi,p) \qquad \Rightarrow \qquad p(\psi,E)=\sqrt{2m\left(E-V(\psi)\right)}$$

$$I(E)=\frac{1}{2\pi}\int_T d\psi\ p(\psi,E)$$

Action-angle variables for the linear (i.e. harmonic) oscillator:

$$H(q,p,t)=\frac{1}{2}\left(\frac{p^2}{m}+m\omega^2q^2\right)$$

$$q=\sqrt{\frac{2I}{m\omega}}\sin\theta, \quad p=\left[2m\omega I-m\omega^2\frac{2I}{\omega}\sin^2\theta\right]^{1/2}=\sqrt{2m\omega I}\cos\theta$$

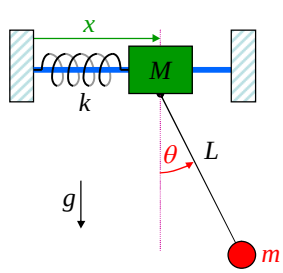
Small Oscillations: Summary and Example

Introduction: A persistent challenge in mechanical engineering is designing machines that preserve their integrity, and are compatible with their surroundings. In other words, they don't fall apart, and don't jump around too much when they operate. Such machines are designed to restrain unwanted motion, but this can never be fully achieved. This very restraint necessarily leads to "small" oscillations. To control them, one must first understand them. We describe here undamped small oscillations. Real machines include damping, but to design a damping system, one must understand the free motion one is trying to stop, and especially the frequency of any oscillations. Such free motion is the topic of small oscillations.

Overview: Small oscillations are the motions of a small perturbation of a system from a stable equilibrium point. The derivatives of the energies of an arbitrary system are evaluated at the equilibrium point, and determine a set of constants (the **M** and **K** matrices), which are independent of any small displacement. This set of constants then defines a linear system (linear in force, quadratic in energy, as usual). Within this linear(ized) system with constant coefficients, we choose small displacements as our coordinates, and then solve the equations of motion to find that they are oscillatory. We will see that the EOM is a matrix generalization of the 1D simple harmonic oscillator. Notation is widely varied on this topic.

This section requires a thorough understanding of the 1D simple harmonic oscillator, and a basic understanding of vectors, matrices, eigenvalues, and matrix notation. It may be helpful to understand phasors (see *Funky Electromagnetic Concepts*), and basis (or canonical) transformations.

As an example, consider a mass moving horizontally on a spring, with a pendulum hanging from it:



System with gravity, using two generalized coordinates: x and θ.

Brief Summary: We now summarize the analysis of small oscillations in general, and then work through the above example. Define the coordinate "vector" (really just a list of coordinates):

$$\mathbf{q}(t)\equiv q_i(t)\equiv\left(q_1,q_2,\ldots q_n\right) \qquad \text{the } n \text{ generalized coordinates}$$

Note that $\mathbf{q}(t)$ is not a vector in the sense of a vector space; e.g. in generalized coordinates you can't add two vectors component-by-component. However, it is convenient (and common) to use vector and matrix notation. The energies of the exact system, before we convert to small oscillations, are given by two functions:

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$$KE \equiv T(\mathbf{q};\dot{\mathbf{q}}), \qquad PE \equiv V(\mathbf{q})$$

Note that this does not allow for magnetic forces, which cannot be derived from scalar potentials.

A system at rest at a stable equilibrium point $\mathbf{q}_e$ will remain motionless forever. Recall that a stable equilibrium point requires that $V(\mathbf{q}_e)$ be a local minimum, and hence all the partial derivatives $\partial V/\partial q_i = 0$, and also the n eigenvalues of the 2nd-derivative matrix are all positive (or at least, non-negative):

$$\lambda_\alpha \equiv eigenvalues\ of\ \left.\frac{\partial^2 V}{\partial q_j \partial q_k}\right|_{\mathbf{q}_e}, \quad \lambda_\alpha > 0 \qquad \left(or\ at\ least, \geq 0 \text{ with additional constraints}\right)$$

$$\alpha = 1, 2, \ldots n$$

Small oscillations are perturbations around $\mathbf{q}_e$. We refer to our oscillations as small displacements $\boldsymbol{\eta}$ from $\mathbf{q}_e$:

$$\mathbf{q}(t) = \mathbf{q}_e + \boldsymbol{\eta}(t), \qquad \boldsymbol{\eta}(t) \equiv (\eta_1, \ldots, \eta_n)\ \ all\ small. \qquad Note\ that \qquad \dot{\boldsymbol{\eta}}(t) = \dot{\mathbf{q}}(t)$$

However, many authors (and professors) simply redefine the variables 'q_i' to be the perturbations (small displacements), instead of using η_j. And virtually everyone redefines the potential energy $V(\boldsymbol{\eta})$ to be the offset from the equilibrium energy, i.e. $V(\mathbf{q}) - V(\mathbf{q}_e) \rightarrow V(\boldsymbol{\eta})$. Then, in the perturbation coordinates, $\boldsymbol{\eta}$:

$$KE \equiv T = \frac{1}{2}\dot{\boldsymbol{\eta}}^T \mathbf{M}\dot{\boldsymbol{\eta}} = \frac{1}{2}\sum_{j,k=1}^{n} M_{jk}\dot{\eta}_j\dot{\eta}_k, \qquad PE \equiv V = \frac{1}{2}\boldsymbol{\eta}^T \mathbf{K}\boldsymbol{\eta} = \frac{1}{2}\sum_{j,k=1}^{n} K_{jk}\eta_j\eta_k$$

The kinetic energy is a quadratic form of the generalized velocities, and the potential energy is a quadratic form of the generalized (small displacement) coordinates. (This again reveals the limitation of no magnetic forces.) The above forms show that we can populate the **M** and **K** matrices from the 2nd derivatives of the kinetic and potential energies:

$$(1) \quad M_{jk} = \frac{\partial^2 T(\dot{\boldsymbol{\eta}})}{\partial \dot{\eta}_j \partial \dot{\eta}_k} = \left.\frac{\partial^2 T(\dot{\mathbf{q}})}{\partial \dot{q}_j \partial \dot{q}_k}\right|_{\mathbf{q}_e}, \qquad K_{jk} = \frac{\partial^2 V(\eta)}{\partial \eta_j \partial \eta_k} = \left.\frac{\partial^2 V(\mathbf{q})}{\partial q_j \partial q_k}\right|_{\mathbf{q}_e},$$

which contain only constant terms (i.e., no η).

The whole system of *small* oscillations is based on the displacements $\boldsymbol{\eta}(t)$ being small. Therefore, **M** and **K** have only constant terms (the kinetic and potential energies are Taylor expanded to 2nd order), so the energies are quadratic in the η. Higher order terms are neglected. This insures that:

The **M** and **K** matrices are constants (contain no η or derivatives). They are also symmetric, and positive definite.

Finally, note that since the generalized coordinates have different units (x in m, θ in rad):

The matrix elements of **M** can have different units from each other, as can the elements of **K**.

The equation of motion (EOM) can be found from Euler-Lagrange, and turns out to be:

$$\mathbf{M}\ddot{\boldsymbol{\eta}} = -\mathbf{K}\boldsymbol{\eta} \qquad \text{which looks a lot like the ODE oscillator case:} \qquad m\ddot{x} = -kx$$

which a linear differential equation with constant coefficients in the vector $\boldsymbol{\eta}$. [If we have arrived at the equation of motion somehow, we can populate the **M** and **K** matrices from it. That's usually much more work than taking derivatives of the $T(\boldsymbol{\eta}\text{-dot})$ and $V(\boldsymbol{\eta})$ functions.] We can solve it similarly to the ODE, by converting to Fourier space (or Fourier modes, or simply phasors). Analogously to the ODE case, we expect the result to be sinusoidal oscillations in time. Then we use the complex-valued phasor vector **a** to represent the sinusoids (**a** is *not* acceleration):

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$$\boldsymbol{\eta}(t) = \mathrm{Re}\left\{\mathbf{a}e^{+i\omega t}\right\} \qquad \textit{where} \quad \omega \text{ is yet to be determined}$$

$$\Rightarrow \qquad \frac{d}{dt} \to (i\omega), \qquad \frac{d^2}{dt^2} \to \left(-\omega^2\right), \qquad \textit{and we have}$$

$$-\omega^2\mathbf{M}\mathbf{a} = -\mathbf{K}\mathbf{a}, \quad \textit{or}$$

(2) $\left(\mathbf{K} - \omega^2\mathbf{M}\right)\mathbf{a} = \mathbf{0}_\mathbf{v}$

Note that **a** is a *constant* phasor vector, and the time dependence is exp(+$i\omega t$), not the usual physics convention of "–$i\omega t$". [Aside: instead of a complex phasor, we could have simply assumed that the solution has the form

$$\boldsymbol{\eta}(t) = \begin{bmatrix} A_1\cos\left(\omega t + \delta_1\right) \\ A_2\cos\left(\omega t + \delta_2\right) \\ \vdots \\ A_n\cos\left(\omega t + \delta_n\right) \end{bmatrix},$$

which yields the same transformation for the 2nd derivative, $\partial^2/dt^2 \to -\omega^2$, but the later algebra is simpler with the complex form.]

Now our equation for the complex vector **a**, eq. (2), can be viewed as a system of n linear equations in n unknowns: the components a_j. It has a non-trivial solution only if the determinant:

$$\left|\mathbf{K} - \omega^2\mathbf{M}\right| = 0 \qquad \textit{or equivalently} \qquad \left|\omega^2\mathbf{M} - \mathbf{K}\right| = 0$$

Since we will be solving for ω^2, we avoid many minus signs by using the form on the right, above. This determinant is an nth order polynomial in ω^2, so it yields n discrete values of ω^2, which will all be positive. For each ω^2, we choose the positive square root $+\omega$, since the negative root simply duplicates the solution of the positive root. Thus:

There are n discrete values of ω, all real, for each of which a solution to the system of small displacements exists.

[F&W p??] say the ω can be degenerate. If so, then we can construct orthogonal eigenvectors in the usual way, such as Graham-Schmidt orthogonalization. The set of ω are the **generalized eigenvalues** of the **generalized eigenvector** equation, eq. (2) above. [If **M** were the identity, it would be a *standard* eigenvector equation.] We solve it by simply solving the set of simultaneous equations, with ordinary algebra. Each frequency, ω_α, yields a distinct phasor eigenvector, $\mathbf{a}_\alpha$, which represents a solution to the small oscillations:

$$\boldsymbol{\eta}_\alpha(t) = \mathrm{Re}\left\{\mathbf{a}_\alpha e^{+i\omega_\alpha t}\right\} \qquad \alpha = 1, 2, \ldots n$$

Thus we have n linearly independent solutions to the equations of motion, $\boldsymbol{\eta}_\alpha(t)$, each of which is a complete set of n coordinate functions of time. Recall that eigenvectors are only defined up to a multiplicative constant.

Now the matrices **M** and **K** are symmetric (i.e., self-adjoint) and positive-definite, which implies the eigenvectors **a** can be chosen to be purely real. This means for a single frequency ω_α (i.e., for a single **mode**), all the coordinates $\eta_i(t)$ cross their zero-points at the same time, i.e. they are all either in-phase or exactly out-of-phase.

Now eq. (2) is linear, so any multiple of a solution is also a solution, and any linear combination of solutions is a solution. This means we have:

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$$\boldsymbol{\eta}_\alpha(t) = A_\alpha \mathbf{a}_\alpha \cos\left(\omega_\alpha t + \delta_\alpha\right) \qquad \textit{where} \quad \mathbf{a} \textit{ is real,} \qquad \textit{and}$$

$$\boldsymbol{\eta}(t) = \sum_{\alpha=1}^{n} A_\alpha \mathbf{a}_\alpha \cos\left(\omega_\alpha t + \delta_\alpha\right) \qquad \textit{where} \quad \boldsymbol{\eta}(t) \textit{ is the most general solution}$$

The $2n$ free parameters A_α and δ_α are the amplitudes and phases of the oscillations, and can be chosen to satisfy $2n$ "boundary conditions" (really, any two constraints on the solution, be they at the boundary or not). In such a combination of modes of oscillation, the coordinates $\eta_i(t)$ no longer cross their zero-points simultaneously.

The modes of oscillation act independently of each other.

Therefore, it is often useful to transform to a set of n coordinates where each coordinate represents a single mode of oscillation. By simply rearranging the above general solution for $\boldsymbol{\eta}(t)$, we can define **normal coordinates** $\xi_\alpha(t)$ such that:

$$\boldsymbol{\eta}(t) = \sum_{\alpha=1}^{n} A_\alpha \mathbf{a}_\alpha \cos\left(\omega t + \delta_\alpha\right) \equiv \sum_{\alpha=1}^{n} \xi_\alpha(t) \mathbf{a}_\alpha \qquad \textit{where} \quad \xi_\alpha(t) \equiv A_\alpha \cos\left(\omega t + \delta_\alpha\right).$$

(Such a transformation often comes up in quantum mechanics, because the resulting normal coordinates are simple harmonic oscillators, and are quantized as such, in the usual quantum way.) Continuing classically, though:

If we write the **M** and **K** matrices in the normal coordinates, they will be diagonal, since the modes of oscillation do not interact.

[If you think about it, the constant parts of the normal coordinates, A_α and δ_α, appear to be a set of $2n$ Hamilton-Jacobi coordinates for the linearized system: they fully define the motion of the system, and are constants of the motion.]

Example: We consider the small oscillations of the above example system. By inspection, we note the equilibrium position:

$$\mathbf{q}_e = \begin{bmatrix} x_e \\ \theta_e \end{bmatrix} = \begin{bmatrix} x_e \\ 0 \end{bmatrix} \qquad \textit{where} \quad x_e \text{ is unknown, but it won't matter.}$$

Also by inspection, we write $T(\mathbf{q}\text{-dot})$ and $V(\mathbf{q})$:

$$\begin{aligned}
T(x,\theta) &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x} + \frac{d}{dt}L\sin\theta\right)^2 + \frac{1}{2}m\left(\frac{d}{dt}L\cos\theta\right)^2 \\
&= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x} + L\cos\theta\,\dot{\theta}\right)^2 + \frac{1}{2}mL^2\sin^2\theta\,\dot{\theta}^2 \\
&= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\dot{x}^2 + mL\cos\theta\,\dot{x}\dot{\theta} + \frac{1}{2}mL^2\cos^2\theta\,\dot{\theta}^2 + \frac{1}{2}mL^2\sin^2\theta\,\dot{\theta}^2 \\
&= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\dot{x}^2 + mL\cos\theta\,\dot{x}\dot{\theta} + \frac{1}{2}mL^2\dot{\theta}^2 \qquad (\text{using } \cos^2\theta + \sin^2\theta = 1) \\
V(x,\theta) &= \frac{1}{2}k\left(x - x_e\right)^2 - mgL\cos\theta
\end{aligned}$$

[Aside: For a single particle, in finding the KE, we would have essentially determined the metric tensor field, $g_{jk}(x, \theta)$. At the equilibrium point, the mass times the metric tensor = the mass matrix: $m\, g_{jk}(x_e, \theta_e) = M_{jk}$. Also, while **q** is not a vector, **q**-dot *is* a vector.]

In the potential energy, we chose a reference zero of $\theta = \pi/2$. If we had chosen $\theta = 0$, then the bob's potential energy would have been more complicated: $mgL(1 - \cos\theta)$. Note that since PE is defined only up to an additive constant, we can always discard constant terms. Had we chosen $\theta = 0$, we would have discarded the constant mgL term anyway, and returned to the $V(x, \theta)$ given above.

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We populate the **M** and **K** matrices with eq: (1) above. First **M**:

$$M_{jk} = \left.\frac{\partial^2 T(\dot{\mathbf{q}})}{\partial \dot{q}_j \partial \dot{q}_k}\right|_{\mathbf{q}_e} = \left.\begin{bmatrix} M+m & mL\cos\theta \\ mL\cos\theta & mL^2 \end{bmatrix}\right|_{\mathbf{q}_e} = \begin{bmatrix} M+m & mL \\ mL & mL^2 \end{bmatrix} \qquad where \quad \mathbf{q}_e = \begin{bmatrix} x_e \\ 0 \end{bmatrix}$$

Note that **M** is independent of x_e, since the x-velocity is independent of where the equilibrium point is.

Now **K**:

$$K_{jk} = \left.\frac{\partial^2 V(\mathbf{q})}{\partial q_j \partial q_k}\right|_{\mathbf{q}_e} = \left.\begin{bmatrix} k & 0 \\ 0 & mgL\cos\theta \end{bmatrix}\right|_{\mathbf{q}_e} = \begin{bmatrix} k & 0 \\ 0 & mgL \end{bmatrix} \qquad where \quad \mathbf{q}_e = \begin{bmatrix} x_e \\ 0 \end{bmatrix}$$

Again, **K** is independent of x_e. Note that **M** and **K** are all given constants of the system, as always. We now define our motion as small oscillations $\boldsymbol{\eta}(t)$, with T and V given by the **M** and **K** matrices:

$$T = \frac{1}{2}\dot{\boldsymbol{\eta}}^T \mathbf{M}\dot{\boldsymbol{\eta}} = \frac{1}{2}(\dot{\eta}_x, \dot{\eta}_\theta)\begin{bmatrix} M+m & mL \\ mL & mL^2 \end{bmatrix}\begin{pmatrix} \dot{\eta}_x \\ \dot{\eta}_\theta \end{pmatrix}, \qquad V = \frac{1}{2}\boldsymbol{\eta}^T \mathbf{K}\boldsymbol{\eta} = \frac{1}{2}(\eta_x, \eta_\theta)\begin{bmatrix} k & 0 \\ 0 & mgL \end{bmatrix}\begin{pmatrix} \eta_x \\ \eta_\theta \end{pmatrix}$$

$$and\ the\ EOM: \quad \mathbf{M}\ddot{\boldsymbol{\eta}} = -\mathbf{K}\boldsymbol{\eta} \qquad or \qquad \mathbf{M}\ddot{\boldsymbol{\eta}} + \mathbf{K}\boldsymbol{\eta} = \mathbf{0_v}$$

Since our solution is sinusoidal oscillations, we use a phasor vector **a** (i.e. we "switch to Fourier space") to represent our small displacements $\boldsymbol{\eta}(t)$. Then:

$$(3) \quad \left(\omega^2\mathbf{M} - \mathbf{K}\right)\mathbf{a} = \mathbf{0_v} \qquad \Rightarrow \qquad \begin{bmatrix} \omega^2(M+m) - k & \omega^2 mL \\ \omega^2 mL & \omega^2 mL^2 - mgL \end{bmatrix}\begin{pmatrix} a_x \\ a_\theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This is two equations in the two unknowns a_x and a_θ. For a non-trivial solution to exist, we must have the determinant of the coefficient matrix = 0:

$$\det\left|\begin{bmatrix} \omega^2(M+m) - k & \omega^2 mL \\ \omega^2 mL & \omega^2 mL^2 - mgL \end{bmatrix}\right| = 0$$

To avoid some tedious algebra, let us follow Taylor [Tay ??], and choose some simple numbers for our parameters, in some appropriate units: Let $m = M = 1$, $k = 2$, $L = 1$, and $mg = 1$. Then we have:

$$0 = \det\left|\begin{bmatrix} 2\omega^2 - 2 & \omega^2 \\ \omega^2 & \omega^2 - 1 \end{bmatrix}\right| = 2\left(\omega^2\right)^2 - 4\omega^2 + 2 - \left(\omega^2\right)^2.$$

$$\left(\omega^2\right)^2 - 4\omega^2 + 2 = 0 \qquad \Rightarrow$$

$$\omega^2 = \frac{4 \pm \sqrt{16 - (4\cdot 2)}}{2} \qquad \rightarrow \qquad \omega_1 = \sqrt{2+\sqrt{2}}, \quad and \quad \omega_2 = \sqrt{2-\sqrt{2}}$$

For each frequency, we get a vector **a** = (a_x, a_θ), by solving eq. (3) above. (Without simple numbers, this is tedious). Since eigenvectors are defined only up to a multiplicative constant, or equivalently, since our general solution allows for an arbitrary multiplicative constant, we can choose $a_x = 1$. Then we have:

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$$For\ (\omega_1)^2 = 2+\sqrt{2}: \qquad 2\left(2+\sqrt{2}\right)-2+\left(2+\sqrt{2}\right)a_\theta = 2+2\sqrt{2}+\left(2+\sqrt{2}\right)a_\theta = 0$$

$$a_\theta = -\frac{2+2\sqrt{2}}{2+\sqrt{2}}\cdot\underbrace{\frac{2-\sqrt{2}}{2-\sqrt{2}}}_{rationalize} = -\frac{4+2\sqrt{2}-4}{2} = -\sqrt{2} \qquad and \qquad \mathbf{a}_1 = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$$

$$For\ (\omega_2)^2 = 2-\sqrt{2}: \qquad 2\left(2-\sqrt{2}\right)-2+\left(2-\sqrt{2}\right)a_\theta = 2-2\sqrt{2}+\left(2-\sqrt{2}\right)a_\theta = 0$$

$$a_\theta = -\frac{2-2\sqrt{2}}{2-\sqrt{2}}\cdot\underbrace{\frac{2+\sqrt{2}}{2+\sqrt{2}}}_{rationalize} = -\frac{4-2\sqrt{2}-4}{2} = +\sqrt{2} \qquad and \qquad \mathbf{a}_2 = \begin{pmatrix} 1 \\ +\sqrt{2} \end{pmatrix}$$

For this problem, we do not need to normalize the vectors. The general solution for small oscillations is then:

$$\boldsymbol{\eta}(t) = A_1\cos\left(\omega_1 t+\delta_1\right)\mathbf{a}_1 + A_2\cos\left(\omega_2 t+\delta_2\right)\mathbf{a}_2$$

$$or \qquad \begin{bmatrix} \eta_x(t) \\ \eta_\theta(t) \end{bmatrix} = A_1\cos\left(\omega_1 t+\delta_1\right)\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} + A_2\cos\left(\omega_2 t+\delta_2\right)\begin{pmatrix} 1 \\ +\sqrt{2} \end{pmatrix}$$

where A_1, δ_1, A_2, δ_2 are determined by initial (or other) conditions. Note that the two eigenvectors are orthogonal with respect to the mass matrix (not by a simple dot product). We confirm this using our mass matrix, with the simple values chosen above for m, M, k, L, and g, which we used to find the eigenvectors. The inner product is:

$$\mathbf{a}_1^T\mathbf{M}\mathbf{a}_2 = \left(1,\ -\sqrt{2}\right)\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} = \left(1,\ -\sqrt{2}\right)\begin{pmatrix} 2+\sqrt{2} \\ 1+\sqrt{2} \end{pmatrix} = 0$$

Summary

- Small oscillations are important real-world phenomena.
- They are analyzed as a matrix-vector generalization of the simple harmonic oscillator.
- A system has n natural frequencies, where n is the number of degrees of freedom (degrees of motion).
- Each frequency is a "mode" of oscillation in which the generalized coordinates all oscillate sinusoidally, with the relative amplitudes fixed; the phases can all be taken as 0, with some of the amplitudes negative. In other words, all coordinates of a single mode go through zero simultaneously.
- The general solution is a linear superposition of all n modes, where the zero-crossings are, in general, no longer simultaneous.
- The motion has $2n$ arbitrary constants, which can be used to match $2n$ initial (or other) conditions.

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Summary in notation similar to Fetter & Walecka [F&W]:

Let η = displacements, $\mathbf{m}$ = mass matrix, $\mathbf{v}$ = potential matrix, $\zeta(t)$ = normal coordinates

Eigenfrequencies: $\det\left|\mathbf{v} - \omega^2\mathbf{m}\right| = 0$ [F&W 22.45 p 97]

Eigenvectors: $\left(\mathbf{v} - \omega_s{}^2\mathbf{m}\right)\vec{\rho} = 0$ [F&W 22.46 p 97]

Orthonormality: $\sum_{\lambda=1}^{n}\sum_{\sigma=1}^{n}\rho_\sigma^{(t)} m_{\sigma\lambda}\rho_\lambda^{(s)} = \delta_{st}$ [F&W 22.47 p97]

Solution: [F&W 22.40 p97]:

$$\eta_\sigma(t) = \sum_{s=1}^{n} C^{(s)}\rho_\sigma^{(s)}\cos\left(\omega^{(s)}t + \phi^{(s)}\right), \qquad \sigma = 1,\ldots,n \text{ (coordinate); } s = \text{mode}$$

Modal matrix: $A_{\sigma s} = \rho_\sigma^{(s)} \quad \Rightarrow \quad \mathbf{A} = \begin{bmatrix} \vec{\rho}^{(1)} & \vec{\rho}^{(2)} & \ldots & \vec{\rho}^{(n)} \\ \downarrow & \downarrow & \ldots & \downarrow \\ & & & \end{bmatrix}$ [F&W 22.51 p98]

[F&W 22.55 p99] $\mathbf{A}^T\mathbf{m}\mathbf{A} = \left[\vec{\rho}^{(s)} \rightarrow\right]\mathbf{m}\left[\vec{\rho}^{(s)} \downarrow\right] = \mathbf{1}$ from orthonormality of ρ

[F&W 22.58,59 p99] $\mathbf{A}^T\mathbf{v}\mathbf{A} = \omega_D^2 = \begin{bmatrix} \omega_1{}^2 & & & \\ & \omega_2{}^2 & & \\ & & \ddots & \\ & & & \omega_n{}^2 \end{bmatrix}$

We construct all the solutions to the small-displacement motion from the eigenvectors. The eigenvectors, $\rho^{(s)}$, are constants (not functions of time). However, strictly speaking, the "normal coordinates" are time-varying coefficients of the eigenvectors $\rho^{(s)}$ [F&W p101]:

$$\eta_\sigma(t) = \sum_{s=1}^{n} C^{(s)}\cos(\omega^{(s)}t + \phi^{(s)})\rho_\sigma^{(s)} \equiv \zeta^{(s)}(t)\rho_\sigma^{(s)}, \qquad \textit{where} \quad \zeta^{(s)}(t) \equiv C^{(s)}\cos(\omega^{(s)}t + \phi^{(s)})$$

$$\sigma = 1,\ldots,n \text{ (coordinate); } s = \text{mode}$$

Normal coordinates: $\vec{\zeta}(t) = \mathbf{A}^T\mathbf{m}\vec{\eta}(t), \qquad \vec{\eta}(t) = \mathbf{A}\vec{\zeta}(t)$ [F&W 22.60,61 p 99]

$$L = \frac{1}{2}\sum_{s=1}^{n}\left(\left(\dot{\zeta}^{(s)}\right)^2 - \left(\omega^{(s)}\right)^2\left(\zeta^{(s)}\right)^2\right) \quad \Rightarrow \quad \ddot{\zeta}^{(s)}(t) = -\left(\omega^{(s)}\right)^2\zeta^{(s)}(t) \quad \text{[F\&W 22.65,66 p 100]}$$

$$\vec{\eta}(t) = \sum_{s=1}^{n}\vec{\rho}^{(s)}C^{(s)}\cos\left(\omega^{(s)}t + \phi^{(s)}\right) \equiv \sum_{s=1}^{n}\vec{\rho}^{(s)}\zeta^{(s)}(t) \qquad \text{[F\&W 22.69 p 101]}$$

If you think about it, the constant parts of the normal coordinates, $C^{(s)}$ and $\phi^{(s)}$, appear to be a set of $2n$ Hamilton-Jacobi coordinates for the linearized system: they fully define the motion of the system, and are constants of the motion.

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Chains

$$L = T - V = \frac{m}{2}\sum_{i=1}^{N}\dot{\eta}_i^{\,2} - \frac{k}{2}\sum_{i=1}^{N}\left(\eta_{i+1}-\eta_i\right)^2, \qquad \eta_0 = \eta_{N+1} = 0$$

$$\textit{Assume solution}: \eta_n = Ae^{inl\alpha}e^{-i\omega t}$$

$$m\ddot{\eta}_i - k\left(\eta_{i+1}-\eta_i\right) + k\left(\eta_i - \eta_{i-1}\right) = m\ddot{\eta}_i + 2k\eta_i - k\left(\eta_{i+1}+\eta_{i-1}\right) = 0 \qquad \text{(F\&W 24.1+ p108+)}$$

$$m\ddot{\mu}_i + 2\frac{\tau}{a}\mu_i - \frac{\tau}{a}\left(\mu_{i+1}+\mu_{i-1}\right) = 0 \qquad \left[k \leftrightarrow \frac{\tau}{a},\ \eta \leftrightarrow \mu\right] \qquad \text{(F\&W 24.8 p110)}$$

$$\mathrm{D_N} = \frac{\sin\left(N+1\right)\psi}{\sin\psi} = 0 \quad \Rightarrow \quad \left(N+1\right)\psi = n\pi \quad \Rightarrow \quad \omega_n^{\,2} = \frac{4\tau}{ma}\sin^2\frac{n\pi}{2\left(N+1\right)},\ n = 1,\ldots N$$

(F&W 24.33+ p113+)

$$\omega^2 = \frac{2\tau}{ma}\left(1-\cos ka\right) = \frac{4\tau}{ma}\sin^2\left(\frac{ka}{2}\right), \quad k = wave\#$$ (F&W 24.43+ p115)

$\Rightarrow$ $\omega(k)$ is dispersion relation.

Fixed ends: $k = \dfrac{n\pi}{a(N+1)}, \quad n = 1,\ldots N$ (F&W 24.55+ p117)

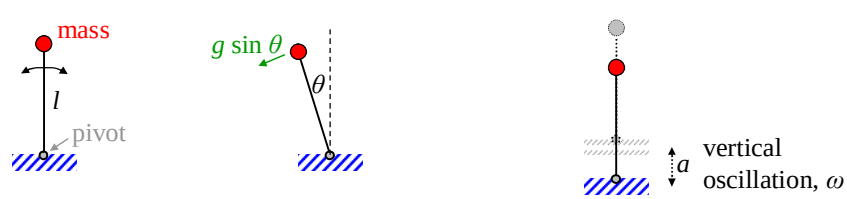
$$\frac{\omega_n}{c} = \frac{2}{a}\sin\left(\frac{n\pi}{l}\frac{a}{2}\right), \qquad l = \left(N+1\right)a, \qquad c = \sqrt{\frac{\tau}{m/a}} = \text{low } \omega \text{ velocity}$$

Alternating chains: *Assume*: $\eta_{2n} = Ae^{i2nl\alpha}e^{-i\omega t}, \qquad \eta_{2n+1} = Be^{i(2n+1)l\alpha}e^{-i\omega t}$

Rapid Perturbations

The effect of rapid perturbations on a slow mechanical system can sometimes be averaged to produce an effective small perturbation on the slow system [P&R p153-7]. As an example, consider this perennial favorite of qualifiers everywhere [P&R p156]:

Question: An inverted pendulum comprises a massless rigid rod with the mass at the top. It has an unstable equilibrium point where the rod is exactly vertical. If we oscillate the whole pendulum fast enough vertically, we can convert the equilibrium to a stable one.



[Note that this is hard to reproduce by balancing a rod on your hand, because the pendulum must accelerate downward faster than gravity to create the equilibrium. Unless you hold the rod, it will leave your hand when you do this.]

Discussion: To analyze this problem, we note that the forces from the forced oscillations must necessarily be *large* compared to the unperturbed force of gravity. If they were never larger than gravity, they could never overcome it, and the pendulum would fall. The fast oscillation results in an effective slow force perturbation because it is *fast*, not because the perturbing force is small.

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The basic principle is that we can approximately separate the fast motion from the slow motion. There will be a small oscillating motion of the pendulum due to the forced fast oscillations. This fast motion is a fast perturbation on the slow motion. The average of the fast motion (position or momentum), over time scales of a period or more, is zero, since it is approximately sinusoidal. However, the fast motion also produces a *slow* term which is proportional to the square of the fast position. This squared term does not average to zero, and produces the perturbation in the slow motion. Therefore, for the slow motion, we average over a full cycle of the fast motion to find the net slow effect.

In short, the slow motion does not (significantly) affect the fast motion, so we solve for the fast motion first. With the fast motion known, we include its effect in the slow motion, and solve for the slow motion.

Solution: Since the forced oscillations induce accelerations (similar to gravity), it is convenient to use the reference frame of the pendulum, and treat the oscillations as an oscillating force of gravity. We use θ, angular displacement from the vertical, as our coordinate.

$$g_{eff} = g - \omega^2 a \cos\omega t \qquad \textit{where} \quad \omega \equiv \text{oscillation frequency, } a \equiv \text{forced amplitude}$$

$$\ddot{\theta} = \frac{g_{eff}\sin\theta}{l} \approx \frac{\left(g - \omega^2 a\cos\omega t\right)}{l}\theta \qquad l \equiv \text{rod length, \& using small angle approximation}$$

We now write θ as the sum of "fast" and "slow" components:

$$\theta(t) \equiv \bar{\theta}(t) + \theta_1(t) \qquad \textit{where} \quad \bar{\theta} \equiv \text{slow motion, } \theta_1 \equiv \text{fast motion}$$

$$\frac{d^2\bar{\theta}}{dt^2} + \frac{d^2\theta_1}{dt^2} = \frac{\left(g - \omega^2 a\cos\omega t\right)}{l}\left(\bar{\theta}(t) + \theta_1(t)\right) = \frac{g\bar{\theta} - \omega^2 a\cos\omega t\bar{\theta} + g\theta_1 - \omega^2 a\left(\cos\omega t\right)\theta_1}{l}$$

We must solve for the fast motion first, since it is independent of the slow motion. We approximate θ_1 as dominated by the form $\cos\omega t$, since it is a response to the forced oscillation. Of the 4 terms on the RHS, we have:

$g\bar{\theta}$	is slow
$-\omega^2 a\cos\omega t\bar{\theta}$	is fast
$+g\theta_1$	is fast, but small, since θ_1 is a perturbation
$-\omega^2 a\left(\cos\omega t\right)\theta_1$	is constant + double fast, since $\theta_1 \sim \cos\omega t \Rightarrow \cos(\omega t)\theta_1 \sim \cos^2\omega t$

For the fast equation, we discard the slow terms, since they are approximately constant over one fast cycle. We also discard the $\cos^2$ term. It can be rewritten as ½ + ½ $\cos 2\omega t$. The constant is slow, and the double-frequency term produces at most a perturbation on our perturbation, which is neglected. Our approximations are only valid if θ_1 is small compared to θ-bar, so only one term survives the fast equation:

$$\frac{d^2\theta_1}{dt^2} = \frac{-\omega^2 a\cos\omega t}{l}\bar{\theta}$$

We solve for θ_1 directly by integrating twice, yielding the cosine form we demanded:

$$\theta_1 = \frac{a\cos\omega t}{l}\bar{\theta}$$

For the slow equation, we average over one fast period. The average of θ_1 is zero, since it is sinusoidal, but the double-fast term produces a slow constant, which is a perturbation on the slow motion:

$$\frac{d^2\bar{\theta}}{dt^2} = \frac{g\bar{\theta} - \left\langle\omega^2 a\left(\cos\omega t\right)\theta_1\right\rangle_{period}}{l} = \left(\frac{g}{l} - \frac{\omega^2 a^2}{2l^2}\right)\bar{\theta} \quad \text{since} \quad \left\langle\omega^2 a\left(\cos\omega t\right)\theta_1\right\rangle_{period} = \frac{1}{2}\frac{\omega^2 a^2}{l}$$

This is the form of a harmonic oscillator, provided

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$$\frac{g}{l} - \frac{\omega^2 a^2}{2l^2} < 0 \qquad or \qquad 2gl < \omega^2 a^2$$

We can therefore increase stability be either increasing the frequency, or increasing the amplitude. Also, shorter pendulums are more stable.

[Aside: this system has a time dependent hamiltonian, and therefore does *not* conserve energy.]

General Derivation

We can derive the general fast-oscillation result for any Hamiltonian system, which gives more insight into the perturbation orders needed. The derivation is straightforward, though somewhat tedious. Following P&R's exposition, our perturbed system comprises a slow hamiltonian, $H_0(q, p)$, plus a large, but fast, perturbing potential, $V(q)\sin \omega t$:

$$H(q,p) = H_0(q,p) + V(q)\sin\omega t \qquad \text{[P\&R 9.65 p153]}$$

$$\textit{Define} \quad Q(t) \equiv \textit{average (slow) position}, \qquad P(t) \equiv \textit{average (slow) momentum}$$

$$\xi(t) \equiv \textit{perturbation position}, \qquad \eta(t) \equiv \textit{perturbation momentum}$$

Hamilton's equations are then

$$\dot{q}(t) = \dot{Q}(t) + \dot{\xi}(t) = \frac{\partial H(q,p)}{\partial p} \qquad \Rightarrow \qquad \dot{Q}(t) + \dot{\xi}(t) = \frac{\partial H\left(Q+\xi, P+\eta\right)}{\partial p}$$

$$\dot{p}(t) = \dot{P}(t) + \dot{\eta}(t) = -\frac{\partial H(q,p)}{\partial q} \qquad \Rightarrow \qquad \dot{P}(t) + \dot{\eta}(t) = -\frac{\partial H\left(Q+\xi, P+\eta\right)}{\partial q}$$

We approximate the derivatives at $(Q+\xi, P+\eta)$ in a 2D Taylor series, but we must first determine our small expansion parameter, and also our orders of expansion parameter to know how many Taylor terms to include. Since the perturbation is fast, ω is large, and $1/\omega$ is small. We may use this as our small expansion parameter. This means that multiplication by $1/\omega$ increases the expansion order by 1, and multiplication by ω decreases the order by one.

We now determine the expansion orders of our dynamic variables, ξ, η, and their derivatives:

$$\xi \sim \sin\omega t \qquad \Rightarrow \qquad \dot{\xi} \sim \omega\xi, \qquad \ddot{\xi} \sim \omega^2\xi$$

$$\eta \sim \dot{\xi} \sim \omega\xi$$

We want the position and momentum perturbations, ξ and η, to be small (at least first order). ξ is one order higher (smaller) than η, so we set η to be first-order. This makes ξ 2nd-order, and the acceleration (ξ-double-dot) zeroth order; this is as we expect since the perturbing force (potential) is comparable to the unperturbed forces (large). Summarizing:

$$\eta \textit{ is 1st order in } 1/\omega \qquad \xi \textit{ is 2nd order} \qquad \dot{\xi} \textit{ is 1st order} \qquad \ddot{\xi} \textit{ is 0th order}$$

Our highest order dynamic variable is ξ, at 2nd order, so we perform our Taylor expansion to 2nd order in $1/\omega$. This requires 1 Taylor term in ξ, and 2 terms in η (note the distinction between H and H_0):

$$\dot{Q} + \dot{\xi} = \frac{\partial H\left(Q+\xi, P+\eta\right)}{\partial p} \approx \frac{\partial H_0\left(Q,P\right)}{\partial p} + \frac{\partial^2 H_0\left(Q,P\right)}{\partial q\,\partial p}\xi + \frac{\partial^2 H_0\left(Q,P\right)}{\partial p^2}\eta + \frac{1}{2}\frac{\partial^3 H_0\left(Q,P\right)}{\partial p^3}\eta^2$$

[corrected P&R 9.73a p155]

The mixed derivative term is missing from [P&R] (it will average out later, or be dropped as 2nd order). The momentum equation is:

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$$\dot{P}+\dot{\eta}=-\frac{\partial H\left(Q+\xi,P+\eta\right)}{\partial q}=-\frac{\partial H_0\left(Q+\xi,P+\eta\right)}{\partial q}-\frac{\partial V(Q+\xi)}{\partial q}\sin\omega t$$

$$=-\frac{\partial H_0\left(Q,P\right)}{\partial q}-\frac{\partial^2 H_0(Q,P)}{\partial q^2}\xi-\frac{\partial^2 H_0\left(Q,P\right)}{\partial p\,\partial q}\eta-\frac{1}{2}\frac{\partial^3 H_0\left(Q,P\right)}{\partial p^2\,\partial q}\eta^2$$

[almost P&R 9.73b, but with correction] $$-\frac{\partial V(Q)}{\partial q}\sin\omega t-\frac{\partial^2 V(Q)}{\partial q^2}\xi\sin\omega t$$

(The 2nd term on the RHS is missing from [P&R 9.73b], but it will disappear in a moment when we average over the fast time-scale.) The first terms in the above position and momentum equations are the slow, unperturbed motion. We can now average over the fast motion, to find the net perturbation effect on the average slow motion. Since both the perturbing position and momentum are zero-mean, the only terms that survive the averaging are those quadratic in the sinusoid, that is η^2 and $\xi \sin \omega t$. This gives:

$$\dot{Q}=\frac{\partial H_0(Q,P)}{\partial p}+\frac{1}{2}\frac{\partial^3 H_0(Q,P)}{\partial p^3}\left\langle\eta^2\right\rangle$$

$$\dot{P}=-\frac{\partial H_0\left(Q,P\right)}{\partial q}-\frac{1}{2}\frac{\partial^3 H_0\left(Q,P\right)}{\partial p^2\,\partial q}\left\langle\eta^2\right\rangle-\frac{\partial^2 V(Q)}{\partial q^2}\left\langle\xi\sin\omega t\right\rangle \qquad \text{[P\&R 9.74a \& b, p155]}$$

To find the fast perturbations ξ and η, we subtract the average motion just above from the full equations 9.73a and b, and retain only the leading order terms. This leaves (recall that η is 1st order, and ξ is 2nd order):

$$\dot{\xi}=\frac{\partial^2 H_0\left(Q,P\right)}{\partial p^2}\eta$$

$$\dot{\eta}=-\frac{\partial V(Q)}{\partial q}\sin\omega t \qquad \left[\frac{\partial V}{\partial q}\text{ is }0^{\text{th}}\text{ order}\right] \qquad \text{[P\&R 9.75 p155]}$$

If the average motion is slow compared to ω, we approximate the derivatives as constant over one period, solve for η directly by integrating, and then use η to solve for ξ:

$$\eta=\frac{1}{\omega}\frac{\partial V(Q)}{\partial q}\cos\omega t, \qquad \xi=\frac{1}{\omega^2}\frac{\partial V(Q)}{\partial q}\frac{\partial^2 H_0\left(Q,P\right)}{\partial p^2}\sin\omega t \qquad \text{[P\&R 9.76 p155]}$$

We can now write the effective equations for the average motion from 9.74 above, using

$$\left\langle\eta^2\right\rangle=\frac{1}{2\omega^2}\left(\frac{\partial V(Q)}{\partial q}\right)^2, \quad \left\langle\xi\sin\omega t\right\rangle=\frac{1}{2\omega^2}\frac{\partial V(Q)}{\partial q}\frac{\partial^2 H_0\left(Q,P\right)}{\partial p^2} \qquad \Rightarrow$$

$$\dot{Q}=\frac{\partial}{\partial p}\left[H_0(Q,P)+\frac{1}{4\omega^2}\left(\frac{\partial V(q)}{\partial q}\right)^2\frac{\partial^2 H_0(Q,P)}{\partial p^2}\right]$$

This looks like Hamilton's position equation for a modified hamiltonian. Amazingly, the same modified hamiltonian satisfies the momentum equation (from above) as well:

$$\dot{P}=-\frac{\partial}{\partial q}\left[H_0\left(Q,P\right)\right]-\frac{1}{4\omega^2}\left(\frac{\partial V(q)}{\partial q}\right)^2\frac{\partial^3 H_0(Q,P)}{\partial p^2\,\partial q}-\frac{\partial^2 V(Q)}{\partial q^2}\frac{1}{2\omega^2}\frac{\partial V(Q)}{\partial q}\frac{\partial^2 H_0\left(Q,P\right)}{\partial p^2}$$

$$=-\frac{\partial}{\partial q}\left[H_0(Q,P)+\frac{1}{4\omega^2}\left(\frac{\partial V(q)}{\partial q}\right)^2\frac{\partial^2 H_0(Q,P)}{\partial p^2}\right]$$

Therefore, the perturbed slow system can be given an effective hamiltonian:

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$$H_{effective}(p,q) = H_0(Q,P) + \frac{1}{4\omega^2}\left(\frac{\partial V(q)}{\partial q}\right)^2 \frac{\partial^2 H_0(Q,P)}{\partial p^2}, \qquad \textit{and}$$

$$\dot{Q} = \frac{\partial}{\partial p} H_{effective} \qquad \dot{P} = -\frac{\partial}{\partial q} H_{effective}$$

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6 Relativistic Mechanics

See *Funky Electromagnetic Concepts* for an introduction to Special Relativity.

Lagrangians for Relativistic Mechanics

At relativistic speeds, the lagrangian is *not* $T - V$. We've already seen that that leads to Newton's laws, which don't work for relativistic speeds. It turns out, the relativistic lagrangian for a charged particle is

$$L(\mathbf{q}, \dot{\mathbf{q}}) = -mc^2\sqrt{1 - v^2/c^2} + \frac{e}{c}\dot{\mathbf{q}} \cdot \mathbf{A} - V(\mathbf{q}) \qquad \text{[LL2 16.4 p48]}$$

$$\textit{where} \quad e \equiv \text{charge of particle}, \qquad V(\mathbf{q}) \equiv \text{potential of particle} = e\phi(\mathbf{q}) \text{ for an electric field}$$

This is written in a particular frame of reference. For non-magnetic interactions, just drop the magnetic term.

Acceleration Without Force

Consider a particle moving in the x-direction, and a force pushing it in the $+y$-direction. The particle accelerates up, but it also *decelerates* in the x-direction, even though there is no force in the x-direction. That's because the acceleration in y increases the particle's magnitude of velocity (speed), and therefore the particle's $\gamma = 1/\sqrt{(1 - v^2/c^2)}$. The *x-momentum* doesn't change, but when γ increases, v_x must *decrease* to keep the same momentum:

$$F_y = \frac{dp_y}{dt} \qquad \Rightarrow \qquad v_y \text{ increases, and } |\mathbf{v}| \text{ and } v^2 \text{ increase.}$$

$$F_x = 0 = \frac{dp_x}{dt} \qquad \Rightarrow \qquad p_x = \gamma m v_x = const. \qquad \gamma \text{ increasing} \qquad \Rightarrow \qquad v_x \text{ decreases!}$$

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7 Appendices

References

Note that web references are moving targets that can become obsolete at any time.

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Glossary

Definitions of common terms:

action — the time integral of the lagrangian.

ansatz — an educated guess that is legitimized later by its results.

arg A — for a complex number A, arg A is the angle of A in the complex plane; i.e., $A = |A|e^{i(\arg A)}$.

cf — "compare to." Abbreviation of Latin "confer."

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classical — non-quantum, or non-relativistic, or both. Classical mechanics is non-quantum, and non-relativistic. Classical quantum mechanics is non-relativistic. Classical Relativity is non-quantum.

contrapositive — The contrapositive of the statement "If A then B" is "If not B then not A." The contrapositive is equivalent to the statement: if the statement is true (or false), the contrapositive is true (or false). If the contrapositive is true (or false), the statement is true (or false).

converse — The converse of the statement "If A then B" is "If B then A". In general, if a statement is true, its converse may be either true or false. The converse is the contrapositive of the inverse, and hence the converse and inverse are equivalent statements.

degrees of freedom — I avoid this term, since it is used conflictingly by other authors. E.g., [F&W] use it to mean both "number of generalized coordinates" [F&W p50b] and also, contradictorily, "number of *independent* degrees of freedom" [F&W p50t].

dynamic variables — Variables that can change in time with the motion of the system. In Lagrangian mechanics, the dynamic variables are $q(t)$ and q-dot(t). In Hamiltonian mechanics, the dynamic variables are $q(t)$ and $p(t)$.

E-L — Euler-Lagrange (pronounced oi'ler-Lah-gronj')

eigen- — German for "natural."

generating function — In Hamiltonian mechanics, a single function of two dynamic variables which defines a specific canonical transformation. There are 4 forms of generating functions. Contrast with "generator".

generator — In Hamiltonian mechanics, a function $G(q, p)$ which defines a continuous family of canonical transformations, parametrized by a continuous parameter (ε), according to $q = q + \varepsilon\, \partial G/\partial p$, and $p = p - \varepsilon\, \partial G/\partial q$. Note that $\varepsilon = 0$ specifies the identity transformation. Contrast with "generating function".

Hamilton's principle — states that the actual trajectory of the motion makes the action of the motion stationary (*not* necessarily minimum) with respect to small variations in the trajectory.

hamiltonian — $H(p,q) \equiv \left(\sum_{n=1}^{\#\,coordinates} p\dot{q} \right) - L(q,\dot{q})$ *where* $L \equiv$ lagrangian of the system

holonomic constraints — can be written as $f_j(x_1,...,x_n,t) = c_j$ =constant , with no velocities.

inverse — The inverse of the statement "If A then B" is "If not A then not B." In general, if a statement is true, its inverse may be either true or false. The inverse is the contrapositive of the converse, and hence the converse and inverse are equivalent statements.

lagrangian — (here not capitalized, as [F&W], though some authors capitalize it [Gol].) The function whose Euler-Lagrange equations of motion are the actual equations of motion. Equivalently, the function whose time integral is the action of motion.

LEM — Lagrange equations of motion. Syn: E-L equations of motion.

path — the locus of points through configuration space of the motion. The path does not include any time information (*when* the system was at any point). Compare to "trajectory".

PT — perturbation theory.

RHS — Right hand side (of an equation).

trajectory — the configuration of the system as a function of time, $q_i(t)$. Compare to "path".

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Formulas

$\sin(a+b) = \sin a \cos b + \cos a \sin b$ $\cos(a+b) = \cos a \cos b - \sin a \sin b$

$L\left(q_i, \dot{q}_i, t\right) = T(q_i, \dot{q}_i, t) - V(q_i, \dot{q}_i, t) + \frac{e}{c}\dot{q}_i \bullet A_i,$ (general non-relativistic lagrangian)

$i = 1, \ldots \#\text{ coordinates}$.

$H(q_i, p_i, t) \equiv \left(\sum_{i=1}^{\#\text{ coordinates}} p_i \dot{q}_i \right) - L(q_i, \dot{q}_i, t), \quad where \quad p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$ [F&W 20.12 p79b].

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